

Continuity of convolution of test functions on Lie groups

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Abstract

For a Lie group G , we show that the map $C_c^\infty(G) \times C_c^\infty(G) \rightarrow C_c^\infty(G)$, $(\gamma, \eta) \mapsto \gamma * \eta$ taking a pair of test functions to their convolution is continuous if and only if G is σ -compact. More generally, consider $r, s, t \in \mathbb{N}_0 \cup \{\infty\}$ with $t \leq r + s$, locally convex spaces E_1, E_2 and a continuous bilinear map $b: E_1 \times E_2 \rightarrow F$ to a complete locally convex space F . Let $\beta: C_c^r(G, E_1) \times C_c^s(G, E_2) \rightarrow C_c^t(G, F)$, $(\gamma, \eta) \mapsto \gamma *_b \eta$ be the associated convolution map. The main result is a characterization of those (G, r, s, t, b) for which β is continuous. Convolution of compactly supported continuous functions on a locally compact group is also discussed, as well as convolution of compactly supported L^1 -functions and convolution of compactly supported Radon measures.

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Introduction and statement of results

It has been known since the beginnings of distribution theory that the bilinear convolution map $\beta: C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n)$, $(\gamma, \eta) \mapsto \gamma * \eta$ (and even convolution $C^\infty(\mathbb{R}^n)' \times C_c^\infty(\mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n)$) is hypocontinuous [40, p. 167]. However, a proof for continuity of β was only published recently [30, Proposition 2.3]. The second author gave an alternative proof [22], which is based on a continuity criterion for bilinear mappings on locally convex direct sums. Our goal is to adapt the latter method to the case where \mathbb{R}^n is replaced with a Lie group, and to the convolution of vector-valued functions.

Let $b: E_1 \times E_2 \rightarrow F$ be a continuous bilinear map between locally convex spaces such that $b \neq 0$. Let $r, s, t \in \mathbb{N}_0 \cup \{\infty\}$ with $t \leq r + s$. If $r = s = t = 0$, let G be a locally compact group; otherwise, let G be a Lie group. Let λ_G be a left Haar measure on G . If G is discrete, we need not impose any completeness assumptions on F . If G is metrizable and not discrete, we assume that F

is sequentially complete or satisfies the metric convex compactness property (i.e., every metrizable compact subset of F has a relatively compact convex hull). If G is not metrizable (and hence not discrete), we assume that F satisfies the convex compactness property (i.e., every compact subset of F has a relatively compact convex hull); this is guaranteed if F is quasi-complete.¹ These conditions ensure the existence of the integrals needed to define the convolution $\gamma *_b \eta: G \rightarrow F$ of $\gamma \in C_c^r(G, E_1)$ and $\eta \in C_c^s(G, E_2)$ via

$$(\gamma *_b \eta)(x) := \int_G b(\gamma(y), \eta(y^{-1}x)) d\lambda_G(y) \quad \text{for } x \in G. \quad (1)$$

Then $\gamma *_b \eta \in C_c^{r+s}(G, F)$ (Proposition 2.2), enabling us to consider the map

$$\beta: C_c^r(G, E_1) \times C_c^s(G, E_2) \rightarrow C_c^t(G, F), \quad (\gamma, \eta) \mapsto \gamma *_b \eta. \quad (2)$$

The mapping β is bilinear, and it is always hypocontinuous (Proposition 2.7). If G is compact, then β is continuous (Corollary 2.3). If G is an infinite discrete group, then β is continuous if and only if G is countable and b ‘admits product estimates’ (Proposition 6.1), in the following sense:

Definition. Let $b: E_1 \times E_2 \rightarrow F$ be a continuous bilinear map between locally convex spaces. We say that b *admits product estimates* if, for each double sequence $(p_{i,j})_{i,j \in \mathbb{N}}$ of continuous seminorms on F , there exists a sequence $(p_i)_{i \in \mathbb{N}}$ of continuous seminorms on E_1 and a sequence $(q_j)_{j \in \mathbb{N}}$ of continuous seminorms on E_2 , such that

$$(\forall i, j \in \mathbb{N}) (\forall x \in E_1) (\forall y \in E_2) \quad p_{i,j}(b(x, y)) \leq p_i(x)q_j(y). \quad (3)$$

Having dealt with compact groups and discrete groups, only one case remains:

Theorem A. *If G is neither discrete nor compact, then the convolution map β from (2) is continuous if and only if all of (a), (b) and (c) are satisfied:*

- (a) G is σ -compact;
- (b) If $t = \infty$, then also $r = s = \infty$;
- (c) b admits product estimates.

¹See [42] for a discussion of these properties.

We mention that (c) is automatically satisfied whenever both E_1 and E_2 are normable [23, Corollary 4.2]. As a consequence, for normable E_1 , E_2 and a Lie group G , the convolution map $\beta: C_c^\infty(G, E_1) \times C_c^\infty(G, E_2) \rightarrow C_c^\infty(G, F)$ is continuous if and only if G is σ -compact. In particular, the convolution map $C_c^\infty(G) \times C_c^\infty(G) \rightarrow C_c^\infty(G)$ is continuous for each σ -compact Lie group G (as first established in the unpublished thesis [10], by a different reasoning), but fails to be continuous if G is not σ -compact.

Further examples of bilinear maps admitting product estimates can be found in [23]. For instance, the convolution map $C^\infty(G) \times C^\infty(G) \rightarrow C^\infty(G)$ admits product estimates whenever G is a compact Lie group. Of course, not every continuous bilinear map does admit product estimates, e.g., the multiplication map $C^\infty[0, 1] \times C^\infty[0, 1] \rightarrow C^\infty[0, 1]$ [23, Example 5.2]. In particular, this gives us an example of a topological algebra A such that the associated convolution map $C_c^\infty(\mathbb{R}, A) \times C_c^\infty(\mathbb{R}, A) \rightarrow C_c^\infty(\mathbb{R}, A)$ is discontinuous. It is also interesting that the convolution map $C_c^\infty(\mathbb{R}) \times C_c^0(\mathbb{R}) \rightarrow C_c^\infty(\mathbb{R})$ is discontinuous (as condition (b) from Theorem A is violated here). This had not been recorded yet in the works [30] and [22] devoted to $G = \mathbb{R}^n$.

Irrespective of locally compactness, we have some information concerning convolution on the space $M_c(G) = \varinjlim M_K(G)$ of compactly supported complex Radon measures on a Hausdorff topological group G . Recall that a topological space X is called *hemicompact* if $X = \bigcup_{n=1}^\infty K_n$ with compact subsets $K_1 \subseteq K_2 \subseteq \dots$ of X , such that each compact subset $K \subseteq X$ is contained in some K_n . A locally compact space is hemicompact if and only if it is σ -compact. We call a Hausdorff topological group G *spacious* if there exist uncountable subsets $A, B \subseteq G$ such that $\{(x, y) \in A \times B : xy \in K\}$ is finite for each compact subset $K \subseteq G$. A locally compact group is spacious if and only if it is not σ -compact (see Remark 4.5).

Theorem B. *Let G be a Hausdorff group and $\beta: M_c(G) \times M_c(G) \rightarrow M_c(G)$, $(\mu, \nu) \mapsto \mu * \nu$ be the convolution map.*

- (a) *If G is hemicompact, then β is continuous.*
- (b) *If G is spacious, then β is not continuous.*

Thus, for locally compact G , the convolution map β from Theorem B is continuous if and only if G is σ -compact. An analogous conclusion

applies to convolution of compactly supported L^1 -functions on a locally compact group (Corollary 4.6). Hemicompact groups arise in the duality theory of abelian topological groups, because dual groups of abelian metrizable groups are hemicompact, and dual groups of abelian hemicompact groups are metrizable ([2]; see [1], [3], [4], [25] for recent studies of such groups).

We also discuss the convolution map $C_c^r(G, E_1) \times C^s(G, E_2) \rightarrow C^t(G, F)$. It is hypocontinuous, but continuous only if G is compact (Proposition 7.1). As a consequence, neither the action $C_c^\infty(G) \times E \rightarrow E$ (nor the action $C_c^\infty(G) \times E^\infty \rightarrow E^\infty$ on the space of smooth vectors) associated to a continuous action $G \times E \rightarrow E$ of a Lie group G on a Fréchet space E need to be continuous (contrary to a claim recently made [17, pp.667–668]). In fact, if G is \mathbb{R} and $\mathbb{R} \times C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ the translation action, then $C_c^\infty(\mathbb{R})$ acts on $C^\infty(\mathbb{R})$ by the convolution map, which is discontinuous by Proposition 7.1 (or the independent study [33]). For details, we refer the reader to [24, Proposition A].

The (G, r, s, t, b) for which β admits product estimates are also known [23].

For recent studies of convolution of vector-valued distributions, we refer to [5], [6] and the references therein. Larcher [33] gives a systematic account of the continuity properties of convolution between classical spaces of scalar-valued functions and distributions on \mathbb{R}^n , and proves discontinuity in some cases in which convolution was previously considered continuous by some authors (like [14] and [41]).

Structure of the article. Sections 1 to 3 are of a preparatory nature and provide basic notation and facts which are similar to familiar special cases and easy to take on faith. Because no direct references are available in the required generality, we do not omit the proofs (which follow classical ideas), but relegate them to an appendix (Appendix C). Appendices A and B compile further preliminaries concerning vector-valued integrals and hypocontinuous bilinear maps. On this footing, our results are established in Sections 4 to 7.

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1 Preliminaries and notation

In this section, we compile notation and basic facts concerning spaces of vector-valued C^r -functions. The proofs are given in Appendix C.

Basic conventions. We write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By a *locally convex space*, we mean a Hausdorff locally convex real topological vector space. If E is such a space, we write E' for the space of continuous linear functionals on E . A map between topological spaces is called a *topological embedding* if it is a homeomorphism onto its image. If E is vector space and p a seminorm on E , we define $B_r^p(x) := \{y \in E : p(y - x) < r\}$ and $\overline{B}_r^p(x) := \{y \in E : p(y - x) \leq r\}$ for $x \in E$ and $r > 0$. If X is a set and $\gamma : X \rightarrow E$ a map, we let $\|\gamma\|_{p,\infty} := \sup_{x \in X} p(\gamma(x))$. If $(E, \|\cdot\|)$ is a normed space and $p = \|\cdot\|$, we write $\|\gamma\|_\infty$ instead of $\|\gamma\|_{p,\infty}$, and $B_r^E(x)$ instead of $B_r^p(x)$. Apart from $\rho d\mu$, we shall also write $\rho \odot \mu$ for measures with a density. The manifolds considered in this article are finite-dimensional, but not necessarily σ -compact or paracompact (unless the contrary is stated). The Lie groups considered are finite-dimensional, real Lie groups.

Vector-valued C^r -functions. Let E and F be locally convex spaces, $U \subseteq E$ an open set and $r \in \mathbb{N}_0 \cup \{\infty\}$. Then a map $\gamma : U \rightarrow F$ is called C^r if it is continuous, the iterated directional derivatives $d^{(j)}\gamma(x, y_1, \dots, y_j) := (D_{y_j} \cdots D_{y_1} \gamma)(x)$ exist for all $j \in \mathbb{N}$ such that $j \leq r$, $x \in U$ and $y_1, \dots, y_j \in E$, and, moreover, each of the maps $d^{(j)}\gamma : U \times E^j \rightarrow F$ is continuous. See [34], [35], [28], [18], [27] for the theory of such functions (in varying degrees of generality as regards E and F). If $E = \mathbb{R}^n$, then a vector-valued function γ as before is C^r if and only if the partial derivatives $\partial^\alpha \gamma : U \rightarrow F$ exist and are continuous, for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq r$ (see, e.g., [27]). Since compositions of C^r -maps are C^r , it makes sense to consider C^r -maps from C^r -manifolds to locally convex spaces. If M is a C^1 -manifold and $\gamma : M \rightarrow E$ a C^1 -map to a locally convex space, we write $d\gamma$ for the second component of the tangent map $T\gamma : TM \rightarrow TE \cong E \times E$. If X is a vector field on M , we define

$$D_X(\gamma) := X.\gamma := d\gamma \circ X. \quad (4)$$

Function spaces and their topologies. Let $r \in \mathbb{N}_0 \cup \{\infty\}$ now and E be a locally convex space. If $r = 0$, let M be a (Hausdorff) locally compact space, and equip the space $C^0(M, E) := C(M, E)$ of continuous E -valued

functions on M with the compact-open topology given by the seminorms

$$\|\cdot\|_{p,K}: C(M, E) \rightarrow [0, \infty[, \quad \gamma \mapsto \|\gamma|_K\|_{p,\infty},$$

for K ranging through the compact subsets of M , and p through the continuous seminorms on E . If $(E, \|\cdot\|_E)$ is a normed space, we abbreviate $\|\cdot\|_K := \|\cdot\|_{\|\cdot\|_E, K}$. To harmonize notation, write $T^0 M := M$ and $d^0 \gamma := \gamma$ for $\gamma \in C^0(M, E) := C(M, E)$. If $r > 0$, let M be a C^r -manifold. For $k \in \mathbb{N}$ with $k \leq r$, set $T^k M := T(T^{k-1} M)$ and $d^k \gamma := d(d^{k-1} \gamma): T^k M \rightarrow E$ for C^k -maps $\gamma: M \rightarrow E$. Thus $T^1 M = TM$ and $d^1 \gamma = d\gamma$. Equip $C^r(M, E)$ with the initial topology with respect to the maps $d^k: C^r(M, E) \rightarrow C(T^k M, E)$ for $k \in \mathbb{N}_0$ with $k \leq r$, where $C(T^k(M), E)$ is equipped with the compact-open topology. Returning to $r \in \mathbb{N}_0 \cup \{\infty\}$, endow $C_A^r(M, E) := \{\gamma \in C^r(M, E): \text{supp}(\gamma) \subseteq A\}$ with the topology induced by $C^r(M, E)$, for each closed subset $A \subseteq M$. Let $\mathcal{K}(M)$ be the set of compact subsets of M . Give $C_c^r(M, E) := \bigcup_{K \in \mathcal{K}(M)} C_K^r(M, E)$ the locally convex direct limit topology. Since each inclusion map $C_K^r(M, E) \rightarrow C^r(M, E)$ is continuous and linear, also the linear inclusion map $C_c^r(M, E) \rightarrow C^r(M, E)$ is continuous. Since $C^r(M, E)$ is Hausdorff, this implies that also $C_c^r(M, E)$ is Hausdorff. We abbreviate $C^r(M) := C^r(M, \mathbb{R})$, $C_K^r(M) := C_K^r(\mathbb{R})$ and $C_c^r(M) := C_c^r(M, \mathbb{R})$.

Facts concerning direct sums. If $(E_i)_{i \in I}$ is a family of locally convex spaces, we shall always equip the direct sum $E := \bigoplus_{i \in I} E_i$ with the locally convex direct sum topology [12]. We often identify E_i with its image in E .

Remark 1.1 If $U_i \subseteq E_i$ is a 0-neighbourhood for $i \in I$, then the convex hull $U := \text{conv}(\bigcup_{i \in I} U_i)$ is a 0-neighbourhood in E , and a basis of 0-neighbourhoods is obtained in this way (as is well-known). If I is countable, then the corresponding ‘boxes’ $\bigoplus_{i \in I} U_i := E \cap \prod_{i \in I} U_i$ form a basis of 0-neighbourhoods in E (cf. [31]). It is clear from this that the topology on E is defined by the seminorms $q: E \rightarrow [0, \infty[$ taking $x = (x_i)_{i \in I}$ to $\sum_{i \in I} q_i(x_i)$, for q_i ranging through the set of continuous seminorms on E_i (because $B_1^q(0) = \text{conv}(\bigcup_{i \in I} B_1^{q_i}(0))$.) If I is countable, we can take the seminorms $q(x) := \max\{q_i(x_i): i \in I\}$ instead (because $B_1^q(0) = \bigoplus_{i \in I} B_1^{q_i}(0)$.)

Lemma 1.2 *Let $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ be families of locally convex spaces and $\lambda_i: E_i \rightarrow F_i$ be a linear map that is topological embedding, for $i \in I$. Then $\bigoplus_{i \in I} \lambda_i: \bigoplus_{i \in I} E_i \rightarrow \bigoplus_{i \in I} F_i$, $(x_i)_{i \in I} \mapsto (\lambda_i(x_i))_{i \in I}$ is a topological embedding.*

Mappings to direct sums.

Lemma 1.3 *Let $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, let M be a locally compact space. If $r > 0$, let M be a C^r -manifold. Let E be a locally convex space, and $(h_j)_{j \in J}$ be a family of functions $h_j \in C_c^r(M)$ whose supports $K_j := \text{supp}(h_j)$ form a locally finite family. Then the map*

$$\Phi: C_c^r(M, E) \rightarrow \bigoplus_{j \in J} C_{K_j}^r(M, E), \quad \gamma \mapsto (h_j \cdot \gamma)_{j \in J}$$

is continuous and linear. If $(h_j)_{j \in J}$ is a partition of unity (i.e., $h_j \geq 0$ and $\sum_{j \in J} h_j = 1$ pointwise), then Φ is a topological embedding.

Lemma 1.4 *Let $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, let M be a locally compact space. If $r > 0$, let M be a C^r -manifold. Let E be a locally convex space, and P be a set of disjoint, open and closed subsets of M , such that $(S)_{S \in P}$ is locally finite. Then*

$$\Phi: C_c^r(M, E) \rightarrow \bigoplus_{S \in P} C_c^r(S, E), \quad \gamma \mapsto (\gamma|_S)_{S \in P}$$

is a continuous linear map. If P is a partition of M into open sets, then Φ is an isomorphism of topological vector spaces.

Seminorms arising from frames. If M is a smooth manifold of dimension m , we call a set $\mathcal{F} = \{X_1, \dots, X_m\}$ of smooth vector fields a *frame* on M if $X_1(p), \dots, X_m(p)$ is a basis for $T_p(M)$, for each $p \in M$. If also $\mathcal{G} = \{Y_1, \dots, Y_m\}$ is a frame on M , then there exist $a_{i,j} \in C^\infty(M)$ for $i, j \in \{1, \dots, m\}$ such that $Y_j = \sum_{i=1}^m a_{i,j} X_i$.

Lemma 1.5 *Let M be a smooth manifold, E be a locally convex space, $k, \ell \in \mathbb{N}_0$ and $\mathcal{F}_1, \dots, \mathcal{F}_k$ be frames on M . Let $\gamma: M \rightarrow E$ be a C^k -map such that $X_j \cdots X_1 \gamma \in C^\ell(M, E)$ for all $j \in \mathbb{N}_0$ with $j \leq k$ and $X_i \in \mathcal{F}_i$ for $i \in \{1, \dots, j\}$. Then γ is $C^{k+\ell}$.*

Lemma 1.6 *Let E be a locally convex space, M be a smooth manifold, $r \in \mathbb{N}$ and $\mathcal{F} := (\mathcal{F}_1, \dots, \mathcal{F}_r)$ be an r -tuple of frames on M . Then the usual topology \mathcal{O} on $C^r(M, E)$ coincides with the initial topology $\mathcal{T}_{\mathcal{F}}$ with respect to the maps*

$$D_{X_j, \dots, X_1}: C^r(M, E) \rightarrow C^0(M, E)_{\text{c.o.}}, \quad \gamma \mapsto X_j \cdots X_1 \gamma,$$

where $j \in \{0, \dots, r\}$ and $X_i \in \mathcal{F}_i$ for $i \in \{1, \dots, j\}$. As a consequence, for each closed subset $K \subseteq M$, the topology on $C_K^r(M, E)$ is initial with respect to the maps $C_K^r(M, E) \rightarrow C_K^0(M, E)_{\text{c.o.}}$, $\gamma \mapsto X_j \cdots X_1 \gamma$, where $j \in \{0, \dots, r\}$ and $X_i \in \mathcal{F}_i$ for $i \in \{1, \dots, j\}$.

Definition 1.7 Let G be a Lie group, with identity element 1. Given $g \in G$, we define the left translation map $L_g: G \rightarrow G$, $L_g(x) := gx$ and the right translation map $R_g: G \rightarrow G$, $R_g(x) := xg$. Let \mathcal{B} be a basis of the tangent space $T_1(G)$, and E be a locally convex space. For $v \in \mathcal{B}$, let \mathcal{L}_v be the left-invariant vector field on G defined via $\mathcal{L}_v(g) := T_1(L_g)(v)$, and \mathcal{R}_v the right-invariant vector field given by $\mathcal{R}_v(g) := T_1(R_g)(v)$. Write

$$\mathcal{F}_L := \{\mathcal{L}_v: v \in \mathcal{B}\} \quad \text{and} \quad \mathcal{F}_R := \{\mathcal{R}_v: v \in \mathcal{B}\}.$$

Let $K \subseteq G$ be compact. Given $r \in \mathbb{N}_0 \cup \{\infty\}$, $k, \ell \in \mathbb{N}_0$ with $k + \ell \leq r$, and a continuous seminorm p on E , we define $\|\gamma\|_{k,p}^L$ (resp., $\|\gamma\|_{k,p}^R$) for $\gamma \in C_K^r(G, E)$ as the maximum of the numbers

$$\|X_j \dots X_1 \cdot \gamma\|_{p,\infty},$$

for $j \in \{0, \dots, k\}$ and $X_1, \dots, X_j \in \mathcal{F}_L$ (resp., $X_1, \dots, X_j \in \mathcal{F}_R$). We also define $\|\gamma\|_{k,\ell,p}^{L,R}$ (resp., $\|\gamma\|_{k,\ell,p}^{R,L}$) as the maximum of the numbers

$$\|X_i \dots X_1 \cdot Y_j \dots Y_1 \cdot \gamma\|_{p,\infty},$$

for $i \in \{0, \dots, k\}$, $j \in \{0, \dots, \ell\}$ and $X_1, \dots, X_i \in \mathcal{F}_L$, $Y_1, \dots, Y_j \in \mathcal{F}_R$ (resp., $X_1, \dots, X_i \in \mathcal{F}_R$ and $Y_1, \dots, Y_j \in \mathcal{F}_L$). Then $\|\cdot\|_{k,p}^L$, $\|\cdot\|_{k,p}^R$, $\|\cdot\|_{k,\ell,p}^{L,R}$ and $\|\cdot\|_{k,\ell,p}^{R,L}$ are seminorms on $C_K^r(G, E)$. If $E = \mathbb{R}$ and $p = |\cdot|$, we relax notation and also write $\|\cdot\|_k^L$, $\|\cdot\|_k^R$, $\|\cdot\|_{k,\ell}^{L,R}$ and $\|\cdot\|_{k,\ell}^{R,L}$ instead of $\|\cdot\|_{k,p}^L$, $\|\cdot\|_{k,p}^R$, $\|\cdot\|_{k,\ell,p}^{L,R}$ and $\|\cdot\|_{k,\ell,p}^{R,L}$, respectively. The same symbols will be used for the corresponding seminorms on $C_c^r(G, E)$ (defined by the same formulas). For $\ell \in \mathbb{N}_0$ with $\ell \leq r$, we shall also need the seminorm $\|\cdot\|_{\ell,K}^L$ on $C_c^r(G)$ defined as the maximum of the numbers $\|X_j \dots X_1 \cdot \gamma|_K\|_\infty$ for $j \in \{0, \dots, \ell\}$ and $X_1, \dots, X_j \in \mathcal{F}_L$. For each compact set $A \subseteq G$, we have $\|\gamma\|_{\ell,K}^L \leq \|\gamma\|_\ell^L$ for each $\gamma \in C_A^r(G)$. Hence $\|\cdot\|_{\ell,K}^L$ is continuous on $C_A^r(G)$ for each A and hence continuous on the locally convex direct limit $C_c^r(G)$.

To enable uniform notation in the proofs for Lie groups and locally compact groups, we shall write $\|\cdot\|_{0,p}^L := \|\cdot\|_{0,p}^R := \|\cdot\|_{0,0,p}^{R,L} := \|\cdot\|_{0,0,p}^{L,R} := \|\cdot\|_{p,\infty}$ if p is a continuous seminorm on E and G a locally compact group. If $E = \mathbb{R}$ and $K \subseteq G$ is a compact set, we shall also write $\|\cdot\|_{0,K}^L := \|\cdot\|_K$.

In the situation of Definition 1.7, we have:

Lemma 1.8 *For each $t \in \mathbb{N}_0 \cup \{\infty\}$, compact set $K \subseteq G$ and locally convex space E , the topology on $C_K^t(G, E)$ coincides with the topologies defined by each of the following families of seminorms:*

- (a) *The family of the seminorms $\|\cdot\|_{j,p}^L$, for $j \in \mathbb{N}_0$ such that $j \leq t$ and continuous seminorms p on E ;*
- (b) *The family of the seminorms $\|\cdot\|_{j,p}^R$, for $j \in \mathbb{N}_0$ such that $j \leq t$ and continuous seminorms p on E .*

If $t < \infty$ and $t = k + \ell$, then the topology on $C_K^t(G, E)$ is also defined by the seminorms $\|\cdot\|_{k,\ell,p}^{L,R}$, for continuous seminorms p on E (respectively, by the seminorms $\|\cdot\|_{k,\ell,p}^{R,L}$).

Useful automorphisms. We record several isomorphisms of topological vector spaces, for later use.

Definition 1.9 If G is a group, $\gamma: G \rightarrow E$ a map to a vector space and $g \in G$, we define the left translate $\tau_g^L(\gamma): G \rightarrow E$ and the right translate $\tau_g^R(\gamma): G \rightarrow E$ via $\tau_g^L(\gamma)(x) := \gamma(gx)$ and $\tau_g^R(\gamma)(x) := \gamma(xg)$ for $x \in G$.

Lemma 1.10 *Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and E be a locally convex space. If $r = 0$, let G be a locally compact group; otherwise, let G be a Lie group. Let $g \in G$. Then $\gamma \mapsto \tau_g^L(\gamma)$ defines isomorphisms $C^r(G, E) \rightarrow C^r(G, E)$, $C_K^r(G, E) \rightarrow C_{g^{-1}K}^r(G, E)$ (for $K \subseteq G$ compact) and $C_c^r(G, E) \rightarrow C_c^r(G, E)$ of topological vector spaces. Likewise, $\gamma \mapsto \tau_g^R(\gamma)$ defines isomorphisms $C^r(G, E) \rightarrow C^r(G, E)$, $C_K^r(G, E) \rightarrow C_{Kg^{-1}}^r(G, E)$ (for $K \subseteq G$ compact) and $C_c^r(G, E) \rightarrow C_c^r(G, E)$ of topological vector spaces.*

Lemma 1.11 *For each $\ell \in \mathbb{N}_0$ such that $\ell \leq r$, $\gamma \in C_c^r(G)$, compact subset $K \subseteq G$ and $g \in G$, we have $\|\tau_g^L(\gamma)\|_{\ell, g^{-1}K}^L = \|\gamma\|_{\ell, K}^L$.*

Definition 1.12 If G is a locally compact group, we let λ_G be a Haar measure on G , i.e., a left invariant, non-zero Radon measure (cf. 3.1). We let $\Delta_G: G \rightarrow]0, \infty[$ be the modular function, determined by $\lambda_G(Ex) = \Delta_G(x)\lambda_G(E)$ for all $x \in G$ and Borel sets $E \subseteq G$. It is known that Δ_G is a continuous homomorphism [16, 2.24] (and hence smooth if G is a Lie group). If $\gamma: G \rightarrow E$ is a mapping to a vector space, we define $\gamma^*: G \rightarrow E$ via

$$\gamma^*(x) := \Delta_G(x^{-1})\gamma(x^{-1}).$$

It is clear from the definition that $(\gamma^*)^* = \gamma$.

Lemma 1.13 *Let E be a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r > 0$, let G be a Lie group; if $r = 0$, let G be a locally compact group. Then all of the following maps are isomorphisms of topological vector spaces:*

$$\Theta: C^r(G, E) \rightarrow C^r(G, E), \quad \gamma \mapsto \gamma^*;$$

$$\Theta_K: C_K^r(G, E) \rightarrow C_{K^{-1}}^r(G, E), \quad \gamma \mapsto \gamma^*,$$

for K a compact subset of G ; and

$$\Theta_c: C_c^r(G, E) \rightarrow C_c^r(G, E), \quad \gamma \mapsto \gamma^*.$$

Further facts concerning function spaces. Whenever we prove that mappings to spaces of test functions are discontinuous, the following embedding will allow us to reduce to the case of scalar-valued test functions.

Lemma 1.14 *For each C^r -manifold M (resp., locally compact space M , if $r = 0$), locally convex space E and $0 \neq v \in E$, the map*

$$\Phi_v: C_c^r(M) \rightarrow C_c^r(M, E), \quad \Phi_v(\gamma) := \gamma v$$

is linear and a topological embedding (where $(\gamma v)(x) := \gamma(x)v$).

The following related result will be used in Section 6.

Lemma 1.15 *Let $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r > 0$, let M be a C^r -manifold; if $r = 0$, let M be a Hausdorff topological space. Then the bilinear mapping $\Psi_E: C^r(M) \times E \rightarrow C^r(M, E)$, $(\gamma, v) \mapsto \gamma v$ is continuous. If M is locally compact and $K \subseteq M$ compact, then also $\Psi_{K,E}: C_K^r(M) \times E \rightarrow C_K^r(M, E)$, $(\gamma, v) \mapsto \gamma v$ is continuous.*

Lemma 1.16 *Let E be a locally convex space, and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, let M be a locally compact space. If $r > 0$, let M be a C^r -manifold. Then:*

- (a) *For each compact set $K \subseteq M$, there exists a family $(\lambda_i)_{i \in I}$ of continuous linear maps $\lambda_i: E \rightarrow F_i$ to Banach spaces F_i , such that the topology on $C_K^r(M, E)$ is initial with respect to the linear mappings $C_K^r(M, \lambda_i): C_K^r(M, E) \rightarrow C_K^r(M, F_i)$, $\gamma \mapsto \lambda_i \circ \gamma$ for $i \in I$.*

- (b) If M is σ -compact, then there exists a family $(\lambda_i)_{i \in I}$ of continuous linear maps $\lambda_i: E \rightarrow F_i$ to Fréchet spaces F_i , such that the topology on $C_c^r(M, E)$ is initial with respect to the linear mappings $C_c^r(M, \lambda_i): C_c^r(M, E) \rightarrow C_c^r(M, F_i)$, $\gamma \mapsto \lambda_i \circ \gamma$ for $i \in I$.
- (c) If M is paracompact and $B \subseteq C_c^r(M, E)$ is a bounded set, then $B \subseteq C_K^r(M, E)$ for some compact set $K \subseteq M$.

2 Basic facts concerning convolution

Throughout this section, G is a locally compact group, with left Haar measure λ_G , and $b: E_1 \times E_2 \rightarrow F$ a continuous bilinear map between locally convex spaces. As in the previous section, we refer to Appendix C for all proofs. If G is not metrizable, we assume that F satisfies the convex compactness property. If G is metrizable and not discrete, we assume that F is sequentially complete or satisfies the metric convex compactness property. Given $\gamma \in C(G, E_1)$ and $\eta \in C(G, E_2)$ such that γ or η has compact support, we define

$$\gamma *_b \eta: G \rightarrow F, \quad (\gamma *_b \eta)(x) := \int_G b(\gamma(y), \eta(y^{-1}x)) d\lambda_G(y),$$

noting that the E -valued weak integral exists by Lemma A.1 as the map $G \rightarrow F$, $y \mapsto b(\gamma(y), \eta(y^{-1}x))$ is continuous with support in the compact set

$$\text{supp}(\gamma) \cap x(\text{supp}(\eta))^{-1}. \quad (5)$$

In particular,

$$(\gamma *_b \eta)(x) = \int_{\text{supp}(\gamma)} b(\gamma(y), \eta(y^{-1}x)) d\lambda_G(y). \quad (6)$$

If b is understood, we simply write $\gamma * \eta := \gamma *_b \eta$. Consider the inversion map $j_G: G \rightarrow G$, $g \mapsto g^{-1}$. It is well known (see [16, 2.31]) that the image measure $j_G(\lambda_G)$ is of the form

$$j_G(\lambda_G) = \Delta_G(y^{-1}) d\lambda_G(y). \quad (7)$$

Since $y^{-1}x = (x^{-1}y)^{-1} = j_G(L_{x^{-1}}(y))$, we infer $j_G(L_{x^{-1}}(\lambda_G)) = j_G(\lambda_G) = \Delta_G(y^{-1}) d\lambda_G(y)$. Now the Transformation Formula implies that² $(\gamma *_b \eta)(x) =$

²Apply continuous linear functionals and use [7, 17.3 and 19.3].

$$\int_G b \circ (\gamma \circ L_x \circ j_G, \eta) \circ j_G \circ L_{x^{-1}} d\lambda_G = \int_G b \circ (\gamma \circ L_x \circ j_G, \eta) d((j_G \circ L_{x^{-1}})(\lambda_G)) =$$

$$\int_G b \circ (\gamma \circ L_x \circ j_G, \eta) d(\Delta_G(\cdot^{-1}) \odot \lambda_G) = \int_G b(\gamma(xz^{-1}), \eta(z)) \Delta_G(z^{-1}) d\lambda_G(z).$$
 Thus

$$(\gamma *_b \eta)(x) = \int_G b(\gamma(xz^{-1}), \Delta_G(z^{-1})\eta(z)) d\lambda_G(z) \quad (8)$$

for all γ, η as above and $x \in G$.

Lemma 2.1 *Let $K, L \subseteq G$ be closed sets and K or L be compact. For all $\gamma \in C_K(G, E_1)$, $\eta \in C_L(G, E_2)$, we then have $\gamma *_b \eta \in C_{KL}(G, F)$, and*

$$\text{supp}(\gamma *_b \eta) \subseteq \text{supp}(\gamma) \text{supp}(\eta) \subseteq KL. \quad (9)$$

The bilinear map $\beta: C_K(G, E_1) \times C_L(G, E_2) \rightarrow C_{KL}(G, F)$, $(\gamma, \eta) \mapsto \gamma *_b \eta$ is continuous.

Let G be a Lie group now.

Proposition 2.2 *Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and $K, L \subseteq G$ be closed subsets such that K or L is compact. Then $\gamma *_b \eta \in C_{KL}^{r+s}(G, F)$ for all $\gamma \in C_K^r(G, E_1)$ and $\eta \in C_L^s(G, E_2)$, with*

$$\mathcal{R}_{w_j} \cdots \mathcal{R}_{w_1} \mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1} (\gamma *_b \eta) = (\mathcal{R}_{w_j} \cdots \mathcal{R}_{w_1} \gamma) *_b (\mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1} \eta) \quad (10)$$

for all $i, j \in \mathbb{N}_0$ with $i \leq s$ and $j \leq r$, and all $v_1, \dots, v_i, w_1, \dots, w_j \in T_1(G)$. Moreover, the bilinear map

$$\beta: C_K^r(G, E_1) \times C_L^s(G, E_2) \rightarrow C_{KL}^{r+s}(G, F), \quad (\gamma, \eta) \mapsto \gamma *_b \eta$$

is continuous.

If G is compact, then $C^r(G, E) = C_K^r(G, E)$ with $K := G$, for each $r \in \mathbb{N}_0 \cup \{\infty\}$ and locally convex space E . Hence Proposition 2.2 yields as a special case:

Corollary 2.3 *If G is compact, then the convolution map*

$$\beta: C^r(G, E_1) \times C^s(G, E_2) \rightarrow C^t(G, F), \quad (\gamma, \eta) \mapsto \gamma *_b \eta$$

is continuous, for all $r, s, t \in \mathbb{N}_0 \cup \{\infty\}$ such that $t \leq r + s$. \square

To each continuous bilinear map $b: E_1 \times E_2 \rightarrow F$ as before, we associate a continuous bilinear map $b^\vee: E_2 \times E_1 \rightarrow F$ via

$$b^\vee(x, y) := b(y, x) \quad \text{for } (x, y) \in E_2 \times E_1. \quad (11)$$

Lemma 2.4 *If one of the maps $\gamma \in C^r(G, E_1)$ and $C^s(G, E_2)$ has compact support, then $(\gamma *_b \eta)^* = \eta^* *_b{}^\vee \gamma^*$.*

Lemma 2.5 *Assume that one of the maps $\gamma \in C^r(G, E_1)$ and $C^s(G, E_2)$ has compact support. Let $g \in G$. Then*

- (a) $\tau_g^L(\gamma *_b \eta) = (\tau_g^L(\gamma)) *_b \eta$
- (b) $\tau_g^R(\gamma *_b \eta) = \gamma *_b (\tau_g^R(\eta)).$

Lemma 2.6 *Let (G, r, s, b) be as in the introduction, $K \subseteq G$ be compact, $\gamma \in C_K^r(G, E_1)$, $\eta \in C_c^s(G, E_2)$ and q, p_1, p_2 be continuous seminorms on F, E_1 and E_2 , respectively, such that $q(b(x, y)) \leq p_1(x)p_2(y)$ for all $(x, y) \in E_1 \times E_2$. Let $k, \ell \in \mathbb{N}_0$ with $k \leq r$ and $\ell \leq s$. Then*

$$\begin{aligned} \|\gamma *_b \eta\|_{k,q}^R &\leq \|\gamma\|_{k,p_1}^R \|\eta\|_{p_2,\infty} \lambda_G(K) \\ \|\gamma *_b \eta\|_{\ell,q}^L &\leq \|\gamma\|_{p_1,\infty} \|\eta\|_{\ell,p_2}^L \lambda_G(K) \\ \|\gamma *_b \eta\|_{k,\ell,q}^{R,L} &\leq \|\gamma\|_{k,p_1}^R \|\eta\|_{\ell,p_2}^L \lambda_G(K). \end{aligned}$$

For a definition and necessary background on hypocontinuous bilinear maps, the reader is referred to Appendix B. In Appendix C, we also prove:

Proposition 2.7 *For all (G, r, s, t, b) as in the introduction, the convolution map $\beta: C_c^r(G, E_1) \times C_c^s(G, E_2) \rightarrow C_c^t(G, F)$, $(\gamma, \eta) \mapsto \gamma *_b \eta$ is hypocontinuous.*

3 Facts on measures and their convolution

In this section, we fix our measure-theoretic setting and state basic definitions and facts concerning spaces of complex Radon measure and convolution of complex Radon measures. As before, proofs can be looked up in Appendix C.

3.1 The setting. If X is a Hausdorff topological space, we write $\mathcal{B}(X)$ for the σ -algebra of Borel sets (which is generated by the set of open subsets of X). A positive measure $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ is called a *Borel measure* if $\mu(K) < \infty$ for each compact subset $K \subseteq X$. Following [8], we shall call a Borel measure μ on X a *Radon measure* if μ is inner regular in the sense that $\mu(A) = \sup\{\mu(K): K \subseteq A \text{ compact}\}$ for each $A \in \mathcal{B}(X)$. The support $\text{supp}(\mu)$ of a Radon measure is the smallest closed subset of X such that $\mu(X \setminus \text{supp}(\mu)) = 0$. A complex measure $\mu: \mathcal{B}(X) \rightarrow \mathbb{C}$ is called a *complex Radon measure* on X if the associated total variation measure $|\mu|$ (as in [38, 6.2]) is a (finite) positive Radon measure. In this case, we set $\text{supp}(\mu) := \text{supp}(|\mu|)$. The total variation norm of μ is defined via $\|\mu\| := |\mu|(X)$. We let $M(X)$ be the space of all complex Radon measures on X . Given a compact set $K \subseteq X$, we let $M_K(X)$ be the space of all $\mu \in M(X)$ such that $\text{supp}(\mu) \subseteq K$. It is clear that the restriction map $(M_K(X), \|\cdot\|) \rightarrow (M(K), \|\cdot\|)$ is an isometric isomorphism, and hence $M_K(X) \cong M(K) \cong (C(K)', \|\cdot\|_{\text{op}})$ is a Banach space (using the Riesz Representation Theorem, [38, 6.19]). We give $M_c(X) := \bigcup_K M_K(X)$ the locally convex direct limit topology, and note that it is Hausdorff (being finer than the normable topology arising from the total variation norm). We let $M(X)_+$ be the set of finite positive Radon measures on X , $M_K(X)_+$ be the subset of Radon measures supported in a given compact set $K \subseteq X$, and $M_c(X)_+ := \bigcup_K M_K(X)_+$. If G is a Hausdorff topological group, with group multiplication $m: G \times G \rightarrow G$, we let $\mu \otimes \nu$ be the Radon product measure of $\mu, \nu \in M(G)_+$ (see [8, 2.1.11]). We define $\mu \otimes \nu$ for $\mu, \nu \in M(G)$ via bilinear extension; then $|\mu \otimes \nu| \leq |\mu| \otimes |\nu|$ (see, e.g., [26, (5.4)]). The convolution of $\mu, \nu \in M(G)$ is defined as the measure $\mu * \nu := m_*(\mu \otimes \nu)$ taking $A \in \mathcal{B}(G)$ to $(\mu \otimes \nu)(m^{-1}(A))$ (cf. [8, 2.1.16]). Since $|\mu * \nu| = |m_*(\mu \otimes \nu)| \leq m_*(|\mu \otimes \nu|) \leq m_*(|\mu| \otimes |\nu|) = |\mu| * |\nu|$, one deduces that $\|\mu * \nu\| \leq (|\mu| * |\nu|)(G) = (|\mu| \otimes |\nu|)(G \times G) = |\mu|(G)|\nu|(G) = \|\mu\| \|\nu\|$. We shall use that³

$$\text{supp}(\mu * \nu) \subseteq \text{supp}(\mu) \text{supp}(\nu) \quad \text{for all } \mu, \nu \in M_c(G). \quad (12)$$

Lemma 3.2 *Let X be a Hausdorff topological space and $(A_j)_{j \in J}$ be a family of Borel subsets of X , such that $J_K := \{j \in J: A_j \cap K \neq \emptyset\}$ is finite for each*

³If $z \in G \setminus \text{supp}(\mu) \text{supp}(\nu) =: U$ and $x, y \in G$ with $xy = z$, then $x \notin \text{supp}(\mu)$ or $y \notin \text{supp}(\nu)$. Hence $m^{-1}(U) \cap (\text{supp}(\mu) \times \text{supp}(\nu)) = \emptyset$ and $|\mu * \nu|(U) \leq (|\mu| * |\nu|)(U) = (|\mu| \otimes |\nu|)(m^{-1}(U)) \leq (|\mu| \otimes |\nu|)(G \times (G \setminus \text{supp}(\nu))) + (|\mu| \otimes |\nu|)((G \setminus \text{supp}(\mu)) \times G) = 0$.

compact subset $K \subseteq X$. Then the map

$$\Phi: M_c(X) \rightarrow \bigoplus_{j \in J} M(A_j), \quad \mu \mapsto (\mu|_{B(A_j)})_{j \in J}$$

is continuous and linear.

Recall the notation $\rho \odot \mu$ for $\rho d\mu$.

Lemma 3.3 *Let X be a Hausdorff topological space which is hemicompact, and $K_1 \subseteq K_2 \subseteq \dots$ be compact subsets of X such that each compact subset of X is contained in some K_n . Let $K_0 := \emptyset$. Given $\mu \in M_c(X)$, we have $\mu_n := \mathbf{1}_{K_n \setminus K_{n-1}} \odot \mu \in M_{K_n}(X)$ for each $n \in \mathbb{N}$, and the map*

$$\Phi: M_c(X) \rightarrow \bigoplus_{n \in \mathbb{N}} M_{K_n}(X), \quad \mu \mapsto (\mu_n)_{n \in \mathbb{N}}$$

is linear and a topological embedding.

If X is a locally compact space, $\mu \geq 0$ a Radon measure on X and $K \subseteq X$ a compact set, we define $(L^1(X, \mu), \|\cdot\|_{L^1})$ as usual and let $L_K^1(X, \mu)$ be the set of all $[\gamma] \in L^1(X, \mu)$ vanishing μ -almost everywhere outside K . We equip $L_K^1(X, \mu)$ with the topology induced by $L^1(X, \mu)$, and $L_c^1(X, \mu) := \bigcup_K L_K^1(X, \mu)$ with the locally convex direct limit topology. We abbreviate $L_c^1(G) := L_c^1(G, \lambda_G)$.

Lemma 3.4 *For each locally compact space X and Radon measure $\mu \geq 0$ on X , the map*

$$\Phi: C_c(X) \rightarrow M_c(X), \quad \gamma \mapsto \gamma \odot \mu$$

is continuous and linear, and so is $\Psi: L_c^1(X, \mu) \rightarrow M_c(X)$, $\gamma \mapsto \gamma \odot \mu$.

As is well-known, the definitions of convolution of functions and that of measures are compatible with one another (cf. [16, p. 50]):

Lemma 3.5 *If G is a locally compact group, with left Haar measure λ_G , then $(\gamma \odot \lambda_G) * (\eta \odot \lambda_G) = (\gamma * \eta) \odot \lambda_G$ for all $\gamma, \eta \in C_c(G)$ (and, more generally, for $\gamma, \eta \in L_c^1(G)$). \square*

4 Convolution on non- σ -compact groups

We prove that convolution of test functions on a non- σ -compact group is always discontinuous (Proposition 4.3). Notably, this shows the necessity of condition (a) in Theorem A. It is efficient to discuss convolution of measures in parallel, and all our results concerning it. [11, Lemma] is essential:

Lemma 4.1 *Let I be an uncountable set. Then there exists a function $g: I \times I \rightarrow]0, \infty[$ such that for each $v: I \rightarrow]0, \infty[$, there exist $i, j \in I$ such that $v(i)v(j) > g(i, j)$. \square*

Lemma 4.2 *Let G be a Hausdorff topological group and $W \subseteq M_c(G)$ be a cone (i.e., $[0, \infty[W \subseteq W$), equipped with a topology \mathcal{O} making the map $m_\mu: [0, \infty[\rightarrow W$, $r \mapsto r\mu$ continuous at 0 for each $\mu \in W$. Assume that there exists an uncountable set I and families $(Y_i)_{i \in I}$ and $(Z_i)_{i \in I}$ of Borel sets in G such that*

$$I_K := \{(i, j) \in I \times I: Y_i Z_j \cap K \neq \emptyset\}$$

is finite for each compact subset $K \subseteq G$, and there exist non-zero measures $\mu_i, \nu_i \in M_c(G)_+ \cap W$ such that $\text{supp}(\mu_i) \subseteq Y_i$ and $\text{supp}(\nu_i) \subseteq Z_i$, for all $i \in I$. Then the convolution map

$$\beta: (W, \mathcal{O}) \times (W, \mathcal{O}) \rightarrow M_c(G), \quad (\mu, \nu) \mapsto \mu * \nu$$

is discontinuous (with respect to the usual locally convex direct limit topology on the right hand side).

Proof. After passing to a positive multiple, we may assume that $\|\mu_i\| = \|\nu_i\| = 1$ for all $i \in I$. Let $g: I \times I \rightarrow]0, \infty[$ be as in Lemma 4.1. By Lemma 3.2, the restriction maps combine to a continuous linear mapping $M_c(G) \rightarrow \bigoplus_{(i,j) \in I \times I} M(Y_i Z_j)$. Hence the set

$$S := \{\mu \in M_c(G): (\forall i, j \in I) |\mu|(Y_i Z_j) < g(i, j)\}$$

is an open 0-neighbourhood in $M_c(G)$. We now show that

$$\beta(U \times V) = U * V \not\subseteq S \tag{13}$$

for any 0-neighbourhoods $U \subseteq W$ and $V \subseteq W$. Hence β will be discontinuous at $(0, 0)$. Since U is a 0-neighbourhood and m_{μ_i} is continuous at 0 for $i \in I$,

we find $\varepsilon_i > 0$ such that $\varepsilon_i \mu_i \in U$. Likewise, we find $\theta_j > 0$ such that $\theta_j \nu_j \in V$. By choice of g , there exist $i, j \in I$ such that $\varepsilon_i \theta_j > g(i, j)$. Since $(\varepsilon_i \mu_i) * (\theta_j \nu_j) \in M_c(G)_+$ and $\text{supp}((\varepsilon_i \mu_i) * (\theta_j \nu_j)) \subseteq \text{supp}(\mu_i) \text{supp}(\nu_j) \subseteq Y_i Z_j$, we obtain

$$\begin{aligned} |(\varepsilon_i \mu_i) * (\theta_j \nu_j)|(Y_i Z_j) &= (\varepsilon_i \mu_i * \theta_j \nu_j)(Y_i Z_j) = (\varepsilon_i \mu_i * \theta_j \nu_j)(G) \\ &= \varepsilon_i \theta_j \mu_i * \nu_j = \varepsilon_i \theta_j (\mu_i \otimes \nu_j)(G \times G) \\ &= \varepsilon_i \theta_j \mu_i(G) \nu_j(G) = \varepsilon_i \theta_j > g(i, j). \end{aligned}$$

Hence $(\varepsilon_i \mu_i) * (\theta_j \nu_j) \notin S$, establishing (13). \square

Proof of Theorem B. (a) If G is hemicompact, define Φ for $X := G$ as in Lemma 3.3. For $i, j \in \mathbb{N}$, let $f_{i,j}: M_{K_i}(G) \times M_{K_j}(G) \rightarrow M_{K_i K_j}(G) \subseteq M_c(G)$ be the convolution map. Then $f_{i,j}$ is continuous, because $\|f_{i,j}(\mu, \nu)\| = \|\mu * \nu\| \leq \|\mu\| \|\nu\|$. Abbreviate $S := \bigoplus_{i \in \mathbb{N}} M_{K_i}(X)$. Since each of the spaces $M_{K_i}(X)$ is normable, it follows that the bilinear map $f: S \times S \rightarrow M_c(G)$, $f((\gamma_i)_{i \in \mathbb{N}}, (\eta_j)_{j \in \mathbb{N}}) := \sum_{i,j \in \mathbb{N}} f_{i,j}(\gamma_i, \eta_j)$ is continuous [22, Corollary 2.4]. Hence also the convolution map β on $M_c(G)$ is continuous, as it can be written in the form $\beta = f \circ (\Phi \times \Phi)$.

(b) If G is spacious, then there exist uncountable subsets $A, B \subseteq G$ such that $\{(a, b) \in A \times B: ab \in K\}$ is finite for each compact set $K \subseteq G$. After replacing A and B by subsets whose cardinality is the smallest uncountable cardinal \aleph_1 , we may assume that there exists a bijection $f: A \rightarrow B$. Then $(\{a\})_{a \in A}$ and $(\{f(a)\})_{a \in A}$ are families of (singleton) subsets of G such that $\{(a, a') \in A \times A: \{a\}\{f(a')\} \cap K \neq \emptyset\}$ is finite for each compact subset $K \subseteq G$. Now define $W := M_c(G)$, with its usual topology, and note that the point measures $\mu_a := \delta_a$ at a and $\nu_{a'} := \delta_{f(a')}$ at $f(a')$ on G are contained in $W \cap M_{\{a\}}(G)_+$ and $W \cap M_{\{f(a')\}}(G)_+$, respectively. Thus Lemma 4.2 shows that β is not continuous. \square

Proposition 4.3 *Let (G, r, s, t, b) and $\beta: C_c^r(G, E_1) \times C_c^s(G, E_2) \rightarrow C_c^t(G, F)$ be as in the introduction. If G is not σ -compact, then β is not continuous.*

Our proof of Proposition 4.3 uses a property of non- σ -compact groups:

Lemma 4.4 *Let G be a locally compact group which is not σ -compact, and $U \subseteq G$ be a σ -compact, open subgroup of G . Then there exist disjoint uncountable subsets $A, B \subseteq G$ such that A and B have the same cardinality,*

$$(\forall (a, b), (a', b') \in A \times B) \quad aUb \cap a'Ub' \neq \emptyset \Rightarrow (a, b) = (a', b'), \quad (14)$$

and $(aUb)_{(a,b) \in A \times B}$ is locally finite. \square

Proof. Let ω_1 be the first uncountable ordinal. Fix a well-ordering \preceq on G . We prove the following assertion $P(\theta)$ for all ordinals $\theta \leq \omega_1$, by transfinite induction:

$P(\theta)$: There exist uniquely determined families $(A_\alpha)_{\alpha \leq \theta}$ and $(B_\alpha)_{\alpha \leq \theta}$ of subsets $A_\alpha, B_\alpha \subseteq G$ and a unique family $(f_\alpha)_{\alpha \leq \theta}$ of bijections $f_\alpha: A_\alpha \rightarrow B_\alpha$ such that $A_0 = B_0 = \emptyset$, A_α and B_α are countable for all $\alpha \leq \theta$ such that $\alpha \neq \omega_1$, moreover

$$\begin{aligned} (\forall \alpha \leq \theta) \quad (\alpha + 1 \leq \theta) \Rightarrow \\ \begin{cases} A_{\alpha+1} = A_\alpha \cup \{x_\alpha\} & \text{with } x_\alpha = \min(G \setminus \langle U \cup A_\alpha \cup B_\alpha \rangle) \text{ and} \\ B_{\alpha+1} = B_\alpha \cup \{y_\alpha\} & \text{with } y_\alpha = \min(G \setminus \langle U \cup A_{\alpha+1} \cup B_\alpha \rangle), \end{cases} \end{aligned} \quad (15)$$

$A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ and $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ for all limit ordinals $\alpha \leq \theta$, and $f_\alpha|_{A_\beta} = f_\beta$ for all $\beta \leq \alpha \leq \theta$.

Here, the minima in (15) refer to the chosen well-ordering on G . Because A_α and B_α are countable if $\alpha + 1 \leq \theta$, the group $\langle U \cup A_\alpha \cup B_\alpha \rangle$ is σ -compact and hence not all of G . As a consequence, its complement $G \setminus \langle U \cup A_\alpha \cup B_\alpha \rangle$ is non-empty and the first minimum in (15) makes sense. Similarly, the second minimum makes sense.

To prove $P(\theta)$ by transfinite induction, note that $P(0)$ is satisfied if and only if $A_0 = B_0 = f_0 = \emptyset$.

If θ is a non-zero limit ordinal and $P(\theta')$ holds for all $\theta' < \theta$, write $((A_\alpha^{\theta'})_{\alpha \leq \theta'}, (B_\alpha^{\theta'})_{\alpha \leq \theta'}, (f_\alpha^{\theta'})_{\alpha \leq \theta'})$ for the triple $((A_\alpha)_{\alpha \leq \theta'}, (B_\alpha)_{\alpha \leq \theta'}, (f_\alpha)_{\alpha \leq \theta'})$ that is uniquely determined by $P(\theta')$. If $\theta'' \leq \theta' < \theta$, the uniqueness in $P(\theta'')$ implies that $A_\alpha^{\theta''} = A_\alpha^{\theta'}$, $B_\alpha^{\theta''} = B_\alpha^{\theta'}$ and $f_\alpha^{\theta''} = f_\alpha^{\theta'}$ for all $\alpha \leq \theta''$. For $\alpha < \theta$, choose $\theta' < \theta$ such that $\alpha \leq \theta'$; then $A_\alpha := A_\alpha^{\theta'}$, $B_\alpha := B_\alpha^{\theta'}$, and $f_\alpha := f_\alpha^{\theta'}$ are independent of the choice of θ' (as just observed). We also set $A_\theta := \bigcup_{\alpha < \theta} A_\alpha$, $B_\theta := \bigcup_{\alpha < \theta} B_\alpha$ and $f_\theta := \bigcup_{\alpha < \theta} f_\alpha$. Then $P(\theta)$ holds.

If $\theta = \theta' + 1$, let $A_\alpha := A_\alpha^{\theta'}$, $B_\alpha := B_\alpha^{\theta'}$, and $f_\alpha := f_\alpha^{\theta'}$ for $\alpha \leq \theta'$. Define $A_\theta := A_{\theta'} \cup \{x_{\theta'}\}$ with $x_{\theta'} = \min(G \setminus \langle U \cup A_{\theta'} \cup B_{\theta'} \rangle)$. Also, define $B_\theta := B_{\theta'} \cup \{y_{\theta'}\}$ with $y_{\theta'} = \min(G \setminus \langle U \cup A_\theta \cup B_{\theta'} \rangle)$. Then $P(\theta)$ is satisfied. The inductive proof is complete.

Now, set $A := A_{\omega_1}$ and $B := B_{\omega_1}$. These are uncountable sets, as they can be considered as the disjoint unions $A = \bigcup_{\alpha < \omega_1} \{x_\alpha\}$ and $B = \bigcup_{\alpha < \omega_1} \{y_\alpha\}$. Moreover, $f_{\omega_1}: A \rightarrow B$ is a bijection, and (14) can be inferred from $P(\omega_1)$. In fact, assume that

$$x_\alpha U y_\beta \cap x_\gamma U y_\delta \neq \emptyset.$$

Thus, there exist $u, w \in U$ such that $x_\alpha u y_\beta = x_\gamma w y_\delta$. Let $\theta := \max\{\alpha, \beta, \gamma, \delta\}$. If $\beta = \theta$ and $\delta < \theta$, then $H := \langle U \cup A_{\theta+1} \cup B_\theta \rangle$ would be a subgroup containing U and all of x_α, x_γ and y_δ . Hence $y_\theta = y_\beta = u^{-1}x_\alpha^{-1}x_\gamma w y_\delta \in H$, contradicting (15). Hence $\beta = \theta$ implies $\delta = \theta = \beta$. Thus $x_\alpha u = x_\gamma w$ in this case. If $\alpha > \gamma$, let $I := \langle U \cup A_\alpha \rangle$. Then $u, w, x_\gamma \in I$ and hence also $x_\alpha = x_\gamma w u^{-1} \in I$, contradicting (15). The same argument excludes the case $\alpha < \gamma$, and thus $\alpha = \gamma$.

Likewise, $\delta = \theta$ implies $\beta = \delta$, from which $\alpha = \gamma$ follows as just shown.

If $\beta < \theta$ and $\delta < \theta$, we may assume that $\alpha = \theta$ (the case $\gamma = \theta$ is analogous). If we would have $\gamma < \alpha$, then $H := \langle U \cup A_\alpha \cup B_\alpha \rangle$ would be a subgroup containing $\{u, y_\beta, x_\gamma, w, y_\delta\}$. Hence $x_\alpha = x_\gamma w y_\delta y_\beta^{-1} u^{-1} \in H$, contradicting (15). Thus $\alpha = \gamma$. But then $u y_\beta = w y_\delta$. Without loss of generality $\beta \leq \delta$. If we would have $\beta < \delta$, then $I := \langle U \cup B_\delta \rangle$ would be a subgroup containing $\{u, y_\beta, w\}$. Hence $y_\delta = w^{-1} u y_\beta$ would be in I , contradicting (15). Thus (14) holds.

If $K \subseteq G$ is a compact set, let Φ be the set of all pairs (α, β) with $\alpha, \beta < \omega_1$ such that $x_\alpha U y_\beta \cap K \neq \emptyset$. To see that Φ is finite, let us suppose that Φ was infinite and derive a contradiction. Case 1: Assume that $\Theta := \{\max\{\alpha, \beta\} : (\alpha, \beta) \in \Phi\}$ is finite. Then (1a) the set $C := \{\beta \leq \alpha_0 : (\alpha_0, \beta) \in \Phi\}$ is infinite for some $\alpha_0 < \omega_1$, or (1b) the set $D := \{\alpha \leq \beta_0 : (\alpha, \beta_0) \in \Phi\}$ is infinite for some $\beta_0 < \omega_1$. In case (1a), K meets $x_{\alpha_0} U y_\beta$ for all $\beta \in C$ (which are disjoint sets), and hence the compact set $x_{\alpha_0}^{-1} K$ meets $U y_\beta$ for all $\beta \in C$, and also these sets are disjoint. But the set $U \backslash G$ of all right cosets of U is an open cover of $x_{\alpha_0}^{-1} K$ by disjoint open sets, and hence $\{S \in U \backslash G : x_{\alpha_0}^{-1} K \cap S \neq \emptyset\}$ must be finite, contradiction. In case (1b), K meets $x_\alpha U y_{\beta_0}$ for all $\alpha \in D$ (which are disjoint sets), and hence the compact set $K y_{\beta_0}^{-1}$ meets $x_\alpha U$ for all $\alpha \in D$, and also these sets are disjoint. But the set G/U of all left cosets of U is an open cover of $K y_{\beta_0}^{-1}$ by disjoint open sets, and hence $\{S \in G/U : K y_{\beta_0}^{-1} \cap S \neq \emptyset\}$ must be finite, contradiction. Case 2: Assume that Θ is infinite. For each $\theta \in \Theta$, pick $(\alpha_\theta, \beta_\theta) \in \Phi$ such that $\max\{\alpha_\theta, \beta_\theta\} = \theta$. Also, pick $z_\theta \in K \cap x_{\alpha_\theta} U y_{\beta_\theta}$. Then (2a) $C := \{\theta \in \Theta : \theta = \beta_\theta\}$ is infinite or (2b) the set $D := \{\theta \in \Theta : \theta = \alpha_\theta > \beta_\theta\}$ is infinite. In case (2a), if $\theta, \theta' \in C$ and $\theta < \theta'$, then $x_{\alpha_\theta} U y_{\beta_\theta} \subseteq \langle U \cup A_{\theta'+1} \cup B_{\theta'} \rangle =: H$ and $x_{\alpha_{\theta'}} \in H$ (as $\alpha_{\theta'} \leq \theta'$). Since $y_{\beta_{\theta'}} = y_{\theta'} \notin H$ by (15), we have $H \cap H y_{\beta_{\theta'}} = \emptyset$ and hence $x_{\alpha_\theta} U y_{\beta_\theta} U \cap x_{\alpha_{\theta'}} U y_{\beta_{\theta'}} = \emptyset$, entailing that z_θ and $z_{\theta'}$ lie in different left cosets of U , i.e., $z_\theta U \cap z_{\theta'} U = \emptyset$. Hence K meets infinitely many left cosets of U , contradiction. In case (2b), if $\theta, \theta' \in D$ and $\theta < \theta'$, then $x_{\alpha_\theta} U y_{\beta_\theta} \subseteq$

$\langle U \cup A_{\theta'} \cup B_{\theta'} \rangle =: H$ and $y_{\beta_{\theta'}} \in H$ (as $\beta_{\theta'} < \theta'$). Since $x_{\alpha_{\theta'}} = x_{\theta'} \notin H$ by (15), we have $H \cap x_{\alpha_{\theta'}} H = \emptyset$ and hence $x_{\alpha_{\theta'}} U y_{\beta_{\theta'}} U \cap x_{\alpha_{\theta'}} U y_{\beta_{\theta'}} = \emptyset$, entailing that z_{θ} and $z_{\theta'}$ lie in different left cosets of U . Hence K meets infinitely many left cosets of U , contradiction. \square

Proof of Proposition 4.3. We write β_b in place of β .

As $b \neq 0$, there exist non-zero vectors $v \in E_1$, $w \in E_2$ and $z \in F$ such that $\beta(v, w) = z$. Let $\Phi_v: C_c^r(G) \rightarrow C_c^r(G, E_1)$, $\Phi_w: C_c^s(G) \rightarrow C_c^s(G, E_2)$ and $\Phi_z: C_c^t(G) \rightarrow C_c^t(G, F)$ be the linear topological embeddings from Lemma 1.14. If $c: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(s, t) \mapsto s \cdot t$ is the scalar multiplication, then

$$\beta_b \circ (\Phi_v \times \Phi_w) = \Phi_z \circ \beta_c. \quad (16)$$

Hence β_b will be discontinuous if we can show that β_c is discontinuous. Let $\theta: M_c(G) \times M_c(G) \rightarrow M_c(G)$ be convolution of measures. Let $U \subseteq G$ be a σ -compact open subgroup. As we assume that G is not σ -compact, Lemma 4.4 provides uncountable subsets $A, B \subseteq G$ and a bijection $f: A \rightarrow B$, such that $(aUb)_{(a,b) \in A \times B}$ is a locally finite family of disjoint open subsets of G . Define $Y_a := aU$ and $Z_a := Uf(a)$ for $a \in A$. Then $(Y_a Z_b)_{(a,b) \in A \times A}$ is a locally finite family of disjoint open subsets of G . The map Φ from Lemma 3.4 (applied with $\mu := \lambda_G$) is continuous linear and injective. We endow its image $W := \text{im}(\Phi) \subseteq M_c(G)$ with the topology making Φ a homeomorphism onto W . For all $a \in A$, there exist non-zero functions $g_a \in C_c^r(G)$ and $h_a \in C_c^s(G)$ with $g_a, h_a \geq 0$ pointwise and $\text{supp}(g_a) \subseteq Y_a$, $\text{supp}(h_a) \subseteq Z_a$. Now the hypotheses of Lemma 4.2 are satisfied with $\mu_a := g_a \odot \lambda_G$ and $\nu_a := h_a \odot \lambda_G$. Hence $\theta|_{W \times W}$ is discontinuous. But $\beta_c = \theta \circ (\Phi \times \Phi)$ (see Lemma 3.5), entailing that also β_c is discontinuous. \square

Remark 4.5 A locally compact group G is spacious if and only if it is not σ -compact. In fact, if G is spacious, then the convolution map $\beta: M_c(G) \times M_c(G) \rightarrow M_c(G)$ is discontinuous (see Theorem B (b)), whence G is not hemicompact (by Theorem B (a)) and hence not σ -compact. If G is not σ -compact, then G is spacious, as a consequence of Lemma 4.4.

Corollary 4.6 *For G a locally compact group, the convolution mapping $\beta: L_c^1(G) \times L_c^1(G) \rightarrow L_c^1(G)$ is continuous if and only if G is σ -compact.*

Proof. If G is σ -compact, using the local compactness we find compact subsets $K_n \subseteq G$ such that $G = \bigcup_{n=1}^{\infty} K_n$ and K_n is contained in the interior

of K_{n+1} . Set $K_0 := \emptyset$ and abbreviate $S := \bigoplus_{n \in \mathbb{N}} L_{K_n}^1(G)$. Then the map

$$\Phi: L_c^1(G) \rightarrow S, \quad \Phi(\gamma) := (\mathbf{1}_{K_i \setminus K_{i-1}} \gamma)_{i \in \mathbb{N}}$$

is linear, injective, and continuous (as in the proof of Lemma 3.3, using that $\|\mathbf{1}_{K_i \setminus K_{i-1}} \gamma\|_{L^1} \leq \|\gamma\|_{L^1}$). Since $\|\gamma * \eta\|_{L^1} \leq \|\gamma\|_{L^1} \|\eta\|_{L^1}$, the restriction $f_{i,j}$ of β to $L_{K_i}^1(G) \times L_{K_j}^1(G)$ is continuous for all $i, j \in \mathbb{N}$. As all of the spaces $L_{K_i}^1(G)$ are normable, [22, Corollary 2.4] shows that $f: S \times S \rightarrow L_c^1(G)$, $f((\gamma_i)_{i \in \mathbb{N}}, (\eta_j)_{j \in \mathbb{N}}) := \sum_{i,j \in \mathbb{N}} f_{i,j}(\gamma_i, \eta_j)$ is continuous. Hence $\beta = f \circ (\Phi \times \Phi)$ is continuous as well.

If G is not σ -compact, let $\theta: M_c(G) \times M_c(G) \rightarrow M_c(G)$ be convolution of measures. Let $U \subseteq G$ be a σ -compact open subgroup, $(Y_a)_{a \in A}$, $(Z_a)_{a \in A}$, g_a and $h_a \in C_c(G)$ be as in the proof of Proposition 4.3. The map Ψ from Lemma 3.4 (applied with $\mu := \lambda_G$) is continuous linear and injective. We endow its image $W := \text{im}(\Psi) \subseteq M_c(G)$ with the topology making Ψ a homeomorphism onto W . Now the hypotheses of Lemma 4.2 are satisfied, whence $\theta|_{W \times W}$ is discontinuous and hence also $\beta = \theta \circ (\Psi \times \Psi)$. \square

5 Convolution of C_c^k -maps and C_c^∞ -maps

In this section, we prove the necessity of condition (b) in Theorem A. Thus, we assume that (b) is violated and deduce that $\beta_b := \beta$ is discontinuous. In view of Proposition 4.3, it suffices to show this if (a) is satisfied.

Thus, let G be a σ -compact, non-discrete, non-compact Lie group, and $\mathfrak{g} := T_1(G)$. If $r = \infty$ and $s < \infty$, we have $\beta_b(\gamma, \eta) = (\beta_{b^\vee}(\eta^*, \gamma^*))^*$ for $(\gamma, \eta) \in C_c^\infty(G, E_1) \times C_c^s(G, E_2)$, where $\beta_{b^\vee}: C_c^s(G, E_2) \times C_c^\infty(G, E_1) \rightarrow C_c^\infty(G, F)$ and $*$ stands for the involutions on $C_c^s(G, E_2)$, $C_c^\infty(G, E_1)$ and $C_c^\infty(G, F)$, respectively, which are isomorphisms of topological vector spaces by Lemma 1.13. Hence discontinuity of β_{b^\vee} will entail discontinuity of β_b . It therefore suffices to assume that $r < \infty$ and $s = \infty$ in the rest of the proof.

I show that the convolution map $\beta: C_c^r(G, E_1) \times C_c^\infty(G, E_2) \rightarrow C_c^\infty(G, F)$ is discontinuous. As in the proof of Proposition 4.3, we may assume that $E_1 = E_2 = F = \mathbb{R}$ and that $b: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is multiplication, for the proof of discontinuity. Let $K \subseteq G$ be a compact identity neighbourhood, and $M \subseteq G$ be a relatively compact, open set such that $KK \subseteq M$. There exists a sequence $(x_i)_{i \in \mathbb{N}}$ in G such that $(x_i M)_{i \in \mathbb{N}}$ is locally finite. For each $i \in \mathbb{N}$, let $h_i \in C_c^\infty(G)$ be a function such that $\text{supp}(h_i) \subseteq x_i M$ and $h_i = 1$ on

some neighbourhood of $x_i K K$. Let Ω be the set of all $\gamma \in C_c^\infty(G)$ such that $\|\gamma\|_{i, x_i K K}^L = \|h_i \gamma\|_{i, x_i K K}^L < 1$ for all $i \in \mathbb{N}$ (with notation as in Definition 1.7). Then Ω is an open 0-neighbourhood in $C_c^\infty(G)$ (cf. Lemma 1.3). If β was continuous, then we could find 0-neighbourhoods $V \subseteq C_c^r(G)$ and $W \subseteq C_c^\infty(G)$ such that $\beta(V \times W) \subseteq \Omega$. There exists $s \in \mathbb{N}_0$ and $\tau > 0$ such that $\{\eta \in C_K^\infty(G) : \|\eta\|_s^L \leq \tau\} \subseteq W$. Also, for each $i \in \mathbb{N}$ there exists $\sigma_i > 0$ such that $\{\gamma \in C_{x_i K}^r(G) : \|\gamma\|_r^L \leq \sigma_i\} \subseteq V$. Thus $\beta(\gamma, \eta) \in \Omega$ and hence

$$\|\gamma * \eta\|_{i, x_i K K}^L < 1$$

for all $\gamma \in C_{x_i K}^r(G)$ and $\eta \in C_K^\infty(G)$ such that $\|\gamma\|_r^L \leq \sigma_i$ and $\|\eta\|_s^L \leq \tau$. Hence, using Lemmas 1.11 and 2.5,

$$\|\gamma * \eta\|_{i, K K}^L = \|\tau_{x_i}^{L-1}(\gamma * \eta)\|_{i, x_i K K}^L = \|(\tau_{x_i}^{L-1} \gamma) * \eta\|_{i, x_i K K}^L < 1$$

for all $\gamma \in C_K^r(G)$ and $\eta \in C_K^\infty(G)$ such that $\|\gamma\|_r^L \leq \sigma_i$ and $\|\eta\|_s^L \leq \tau$. But this contradicts the following lemma. \square

Lemma 5.1 *Let G be a non-discrete Lie group, $K \subseteq G$ be a compact identity neighbourhood, and $r, s \in \mathbb{N}_0$. Then the convolution map*

$$(C_K^r(G), \|\cdot\|_r^L) \times (C_K^\infty(G), \|\cdot\|_s^L) \rightarrow C_{KK}^\infty(G) \quad (17)$$

is discontinuous, if one uses the ordinary Fréchet space topology on the right hand side, but merely the two indicated norms on the left.

Proof. Suppose that the map (17) was continuous – we shall derive a contradiction. Let $\phi: P \rightarrow Q \subseteq \mathfrak{g}$ be a chart for G around 1, such that $\phi(1) = 0$, $P = P^{-1}$, $d\phi|_{\mathfrak{g}} = \text{id}_{\mathfrak{g}}$ and $\phi(x^{-1}) = -\phi(x)$ for all $x \in P$ (for example, a logarithmic chart). After shrinking K , we may assume that $K = \phi^{-1}(A)$ for some compact 0-neighbourhood $A \subseteq Q$ with $[-1, 1]A \subseteq A$. In particular, $K \subseteq P$. Let $m > 0$ be the dimension of G , $\lambda_{\mathfrak{g}}$ be a Haar measure on $(\mathfrak{g}, +)$, and $\lambda_{\mathfrak{g}}|_Q$ be its restriction to a measure on $(Q, \mathcal{B}(Q))$. Then the image measure $\phi_*(\lambda_G|_P)$ is of the form $\rho d\lambda_{\mathfrak{g}}|_Q$ with a smooth function $\rho: Q \rightarrow]0, \infty[$. Given $\gamma \in C_K^\infty(G)$, let $\tilde{\gamma} := \gamma \circ \phi^{-1} \in C_A^\infty(Q)$. Then, for all $\gamma, \eta \in C_K^\infty(G)$,

$$(\gamma * \eta)(0) = \int_Q \tilde{\gamma}(y) \tilde{\eta}(-y) \rho(y) d\lambda_{\mathfrak{g}}(y). \quad (18)$$

If Y is a vector field on G and $\theta := d\phi \circ Y \circ \phi^{-1} \in C^\infty(Q, \mathbb{R}^m)$ its representative with respect to the chart ϕ , then

$$(D_Y \cdot \gamma)^\sim = D_\theta \cdot \tilde{\gamma}, \quad (19)$$

where $D_Y(\gamma)$ is as in (4), and $D_\theta \tilde{\gamma} := d\tilde{\gamma} \circ (\text{id}_Q, \theta)$. Choose $n \in 2\mathbb{N}$ so large that $m + r + s + 2 - 2n < 0$. Pick $h \in C_A^\infty(\mathfrak{g})$ such that $h \neq 0$ and $h(x) = h(-x)$ for all $x \in \mathfrak{g}$. There is $v \in A \setminus \{0\}$ such that $h(v) \neq 0$. Then $D_v^n h \neq 0$. To see this, find $c > 1$ such that $] -c, c[v \subseteq Q$ but $] -c, c[v \not\subseteq A$. Then $g:] -c, c[\rightarrow \mathbb{R}$, $t \mapsto h(tv)$ is a compactly supported non-zero function, whence $g^{(n)}(t_0) \neq 0$ for some $t_0 \in] -c, c[$ and thus $D_v^n h(t_0 v) \neq 0$. For $t \in]0, 1]$, define $\tilde{\gamma}_t, \tilde{\eta}_t \in C_{tA}^\infty(Q) \subseteq C_A^\infty(Q)$ via

$$\tilde{\gamma}_t(x) := t^{r+1} h(x/t) \quad \text{and} \quad \tilde{\eta}_t(x) := t^{s+1} h(x/t)$$

for $x \in Q$. Define $\gamma_t, \eta_t \in C_K^\infty(G)$ via $\gamma_t(x) := \tilde{\gamma}_t(\phi(x))$ and $\eta_t(x) := \tilde{\eta}_t(\phi(x))$ if $x \in P$, $\gamma_t(x) := \eta_t(x) := 0$ if $x \in G \setminus K$.

Claim: $\gamma_t \rightarrow 0$ in $C_K^r(G)$ and $\eta_t \rightarrow 0$ in $(C_K^\infty(G), \|\cdot\|_s^L)$ as $t \rightarrow 0$. But $\|\gamma_t * \eta_t\|_{n,n}^{R,L} \rightarrow \infty$ as $t \rightarrow 0$, whence $\gamma_t * \eta_t \not\rightarrow 0$ in $C_{KK}^\infty(G)$.

Therefore the map in (17) is not continuous, contradiction.

To prove the claim, we first note that

$$(D_{\xi_j} \dots D_{\xi_1} \tilde{\gamma}_t)(x) = \sum_{i=1}^j t^{r+1-i} g_i(t, x) \quad (20)$$

for $j \in \mathbb{N}$, $\xi_1, \dots, \xi_j \in C^\infty(Q, \mathfrak{g})$ and $x \in Q$, where

$$g_j(t, x) = d^{(j)} h(x/t, \xi_1(x), \dots, \xi_j(x))$$

and $g_i(t, x)$ for $i < j$ is a sum of terms of the form $d^{(i)} h(x/t, f_1(x), \dots, f_i(x))$ with suitable smooth functions $f_1, \dots, f_i \in C^\infty(Q, \mathfrak{g})$. This can be established by a straightforward induction, using that application of D_ξ to $x \mapsto d^{(i)} h(x/t, f_1(x), \dots, f_i(x))$ yields⁴

$$\begin{aligned} & \frac{1}{t} d^{(i+1)} h(x/t, f_1(x), \dots, f_i(x), \xi(x)) \\ & + d^{(i)} h(x/t, (D_\xi f_1)(x), \dots, f_i(x)) + \dots + d^{(i)} h(x/t, f_1(x), \dots, D_\xi(f_i)(x)) \end{aligned}$$

for $\xi \in C^\infty(Q, \mathfrak{g})$. A similar description (with s in place of r) can be given for $D_{\xi_j} \dots D_{\xi_1} \tilde{\eta}_t$. We find it useful to abbreviate $h^{(i)} := d^{(i)} h$ and⁵

$$\|h^{(i)}\|_{\text{op}, \infty} := \sup\{\|h^{(i)}(y, \cdot)\|_{\text{op}} : y \in \mathfrak{g}\}.$$

⁴Recall that $d^{(i)} h(x, \cdot): E^i \rightarrow F$ is i -linear (see, e.g., [18], [28], [35]).

⁵As usual, for normed spaces $(E_1, \|\cdot\|_1), \dots, (E_i, \|\cdot\|_i)$ and $(F, \|\cdot\|_F)$ and a continuous i -linear map $B: E_1 \times \dots \times E_i \rightarrow F$, we define $\|B\|_{\text{op}}$ as the supremum of $\|B(x_1, \dots, x_i)\|_F$, where $x_j \in E_j$ with $\|x_j\|_j \leq 1$ for $j = 1, \dots, i$.

Note that $h^{(i)}(x/t, f_1(x), \dots, f_i(x))$ vanishes for x outside tA and hence for $x \notin A$, and that its norm is bounded by

$$\|h^{(i)}\|_{\text{op}, \infty} \|f_1\|_{\infty} \cdots \|f_i\|_{\infty},$$

irrespective of t and x . A similar estimate is available for $g_j(t, x)$. Also, $\|\tilde{\gamma}_t\|_{\infty} \leq \|h\|_{\infty} t^{r+1}$. Hence, if $j \in \{0, \dots, r\}$, we can find $C > 0$ such that

$$\|D_{\xi_j} \dots D_{\xi_1} \tilde{\gamma}_t\|_{\infty} \leq \sum_{i=1}^j t^{r+1-i} C \leq jtC.$$

As a consequence,

$$\max_{|\alpha| \leq r} \|\partial^{\alpha} \tilde{\gamma}_t\|_{\infty} \rightarrow 0$$

as $t \rightarrow 0$ and thus $\tilde{\gamma}_t \rightarrow 0$ in $C_A^r(Q)$, entailing that $\gamma_t \rightarrow 0$ in $C_K^r(G) \cong C_K^r(P)$. Likewise, $\eta_t \rightarrow 0$ in $C_K^s(G)$, whose topology can be described by the norm $\|\cdot\|_s^L$, and thus $\eta_t \rightarrow 0$ in $(C_K^{\infty}(G), \|\cdot\|_s^L)$.

Next, let X be the right invariant vector field on G with $X(1) = v$, and Y be the left invariant vector field with $Y(1) = v$. Let $\xi := d\phi \circ X \circ \phi^{-1}$ and $\zeta := d\phi \circ Y \circ \phi^{-1}$ be the local representatives. By (18) and (19),

$$(D_X^n D_Y^n (\gamma_t * \eta_t))(0) = (D_X^n \gamma_t * D_Y^n \eta_t)(0) = \int_Q (D_{\xi}^n \tilde{\gamma}_t)(y) (D_{\zeta}^n \tilde{\eta}_t)(-y) \rho(y) d\lambda_{\mathfrak{g}}(y).$$

Write $(D_{\xi}^n \tilde{\gamma}_t)(x) = \sum_{i=1}^n t^{r+1-i} g_i(t, x)$ and $(D_{\zeta}^n \tilde{\eta}_t)(x) = \sum_{j=1}^n t^{s+1-j} h_j(t, x)$ as in (20). Then

$$(D_X^n D_Y^n (\gamma_t * \eta_t))(0) = t^{m+r+s+2-2n} \left(t^{-m} \int_Q g_n(t, y) h_n(t, -y) \rho(y) d\lambda_{\mathfrak{g}}(y) + R(t) \right)$$

where $R(t)$ is the sum of the terms $t^{2n-i-j} t^{-m} \int_Q g_i(t, y) h_j(t, -y) \rho(y) d\lambda_{\mathfrak{g}}(y)$ with $i, j \in \{1, \dots, n\}$ and $(i, j) \neq (n, n)$. For these (i, j) ,

$$\begin{aligned} & \left| t^{-m} \int_Q g_i(t, y) h_j(t, -y) \rho(y) d\lambda_{\mathfrak{g}}(y) \right| \\ &= \left| t^{-m} \int_Q h^{(i)}(y/t, f_1(y), \dots, f_i(y)) h^{(j)}(-y/t, k_1(-y), \dots, k_j(-y)) \rho(y) d\lambda_{\mathfrak{g}}(y) \right| \\ &= \left| \int_{Q/t} h^{(i)}(z, f_1(tz), \dots, f_i(tz)) h^{(j)}(-z, k_1(-tz), \dots, k_j(-tz)) \rho(tz) d\lambda_{\mathfrak{g}}(z) \right| \\ &= \left| \int_A h^{(i)}(z, f_1(tz), \dots, f_i(tz)) h^{(j)}(-z, k_1(-tz), \dots, k_j(-tz)) \rho(tz) d\lambda_{\mathfrak{g}}(z) \right| \\ &\leq \|h^{(i)}\|_{\text{op}, \infty} \|h^{(j)}\|_{\text{op}, \infty} \|f_1\|_{\infty} \cdots \|f_i\|_{\infty} \|k_1\|_{\infty} \cdots \|k_j\|_{\infty} \|\rho|_A\|_{\infty} \lambda_{\mathfrak{g}}(A) \end{aligned}$$

(using the substitution $y/t = z$ to obtain the second equality), where the final estimate is independent of t . Since $2n - i - j \geq 1$ and thus $t^{2n-i-j} \rightarrow 0$ as $t \rightarrow 0$, we deduce that $R(t) \rightarrow 0$. Similarly, substituting $z = y/t$, we get

$$\begin{aligned}
& t^{-m} \int_Q g_n(t, y) h_n(t, -y) \rho(y) d\lambda_{\mathfrak{g}}(y) \\
&= t^{-m} \int_Q h^{(n)}(y/t, \xi(x), \dots, \xi(x)) h^{(n)}(-y/t, \zeta(-y), \dots, \zeta(-y)) \rho(y) d\lambda_{\mathfrak{g}}(y) \\
&= \int_{Q/t} h^{(n)}(z, \xi(tz), \dots, \xi(tz)) h^{(n)}(-z, \zeta(-tz), \dots, \zeta(-tz)) \rho(tz) d\lambda_{\mathfrak{g}}(z) \\
&= \int_A h^{(n)}(z, \xi(tz), \dots, \xi(tz)) h^{(n)}(-z, \zeta(-tz), \dots, \zeta(-tz)) \rho(tz) d\lambda_{\mathfrak{g}}(z) \quad (21)
\end{aligned}$$

which tends to

$$\begin{aligned}
& \int_A h^{(n)}(z, \xi(0), \dots, \xi(0)) h^{(n)}(-z, \zeta(0), \dots, \zeta(0)) \rho(0) d\lambda_{\mathfrak{g}}(z) \\
&= \int_A h^{(n)}(z, v, \dots, v) h^{(n)}(-z, v, \dots, v) \rho(0) d\lambda_{\mathfrak{g}}(z) \\
&= \rho(0) \int_A (D_v^n h(z))^2 d\lambda_{\mathfrak{g}}(z) =: a > 0
\end{aligned}$$

as $t \rightarrow 0$. Note that the integrand in (21) is continuous for $(t, y) \in [0, 1] \times A$, whence Lemma A.2 applies. Since $R(t) \rightarrow 0$, there exists $\tau \in]0, 1]$ such that $|R(t)| \leq a/2$ for all $t \in]0, \tau]$. Then $(D_X^n D_Y^n (\gamma_t * \eta_t))(0) \geq t^{m+r+s+2-2n} \frac{a}{2}$ for all $t \in]0, \tau]$, which tends to ∞ as $t \rightarrow 0$. Hence also $\|\gamma_t * \eta_t\|_{n,n}^{R,L} \geq |(D_X^n D_Y^n (\gamma_t * \eta_t))(0)|$ goes to ∞ as $t \rightarrow 0$, and the claim is established. \square

6 Convolution on σ -compact groups

In this section, we complete the proof of Theorem A. As we have already seen in Sections 4 and 5 that conditions (a) and (b) of the theorem are necessary for continuity of β , it suffices to consider the case that (a) and (b) are already satisfied, and to show that β is continuous if and only if condition (c) of the theorem is satisfied. In parallel, we shall establish a result concerning discrete groups. To formulate it, let (G, r, s, t, b) be as in the introduction. If G is discrete, then $C_c^p(G, H) = \bigoplus_{g \in G} H =: H^{(G)}$ with the locally convex direct

sum topology, for each $p \in \mathbb{N}_0 \cup \{\infty\}$ and locally convex space H . We show:

Proposition 6.1 *If G is an infinite discrete group, then the map $\beta: E_1^{(G)} \times E_2^{(G)} \rightarrow F^{(G)}$, $\beta(\gamma, \eta) := \gamma *_b \eta$ is continuous if and only if G is countable and $b: E_1 \times E_2 \rightarrow F$ satisfies product estimates.*

We need only prove Proposition 6.1 for countable G (as the discontinuity of β for uncountable G was already established in Proposition 4.3).

Lemma 6.2 *Let G be a σ -compact, non-compact, locally compact group and $V \subseteq G$ be a compact identity neighbourhood. Then there are sequences $(g_i)_{i \in \mathbb{N}}$ and $(h_j)_{j \in \mathbb{N}}$ in G , such that the family $(g_i V h_j V)_{(i,j) \in \mathbb{N}^2}$ is locally finite.*

Proof. Since G is locally compact and σ -compact, there exists a sequence $(K_i)_{i \in \mathbb{N}}$ of compact subsets of G such that $G = \bigcup_{i \in \mathbb{N}} K_i$ and $K_i \subseteq K_{i+1}^\circ$, for all $i \in \mathbb{N}$. We may assume that $K_1 = \emptyset$. It suffices to find sequences $(g_i)_{i \in \mathbb{N}}$ and $(h_j)_{j \in \mathbb{N}}$ in G such that

$$g_i V h_j V \cap K_{i \vee j} = \emptyset \quad (22)$$

for all $i, j \in \mathbb{N}$, where $i \vee j$ denotes the maximum of i and j . Indeed, if $K \subseteq G$ is compact, then $K \subseteq K_n$ for some $n \in \mathbb{N}$ and thus $K \cap g_i V h_j V = \emptyset$ unless $i, j \in \{1, \dots, n-1\}$ (which is a finite set). To find such sequences, we make an arbitrary choice of $g_1, h_1 \in G$. Let $n \in \mathbb{N}$ now and assume that g_i, h_j have been chosen for $i, j \in \{1, \dots, n\}$ such that (22) holds. Then the subset

$$P := \bigcup_{j=1}^n K_{n+1} V^{-1} h_j^{-1} V^{-1}$$

of G is compact. As G is non-compact, we find $g_{n+1} \in G \setminus P$. Also $Q := \bigcup_{i=1}^{n+1} V^{-1} g_i^{-1} K_{n+1} V^{-1}$ is compact, whence we find $h_{n+1} \in G \setminus Q$. Then (22) holds for all $i, j \in \{1, \dots, n+1\}$. We need only check this if $i = n+1$ or $j = n+1$. If $j = n+1$, then $h_j = h_{n+1} \notin Q$ and thus $g_i V h_j V \cap K_{n+1} = \emptyset$. If $j \leq n$ and $i = n+1$, then $g_i = g_{n+1} \notin P$ and hence $g_i V h_j V \cap K_{n+1} = \emptyset$. \square

We shall use the seminorm $\|\cdot\|_{p,L^1}$ on $C_c^0(G, E)$ (where p is a continuous seminorm on E), defined via $\|\gamma\|_{p,L^1} := \int_G p(\gamma(y)) d\lambda_G(y)$. For each compact subset $K \subseteq G$, we have $\|\gamma\|_{p,L^1} \leq \lambda_G(K) \|\gamma\|_{p,\infty}$ for all $\gamma \in C_K^0(G, E)$. Hence $\|\cdot\|_{p,L^1}$ is continuous on $C_K^0(G, E)$ and hence also on $C_c^0(G, E)$.

Necessity of product estimates. Let (G, r, s, t, b) be as in the introduction. Assume that G is not compact, and assume that the conditions (a) and (b) from Theorem A are satisfied.⁶ Also, assume that β is continuous. Pick a relatively compact, open identity neighbourhood $V \subseteq G$. By Lemma 6.2, there are sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ in G such that the family $(V_{i,j})_{(i,j) \in \mathbb{N}^2}$ of open sets $V_{i,j} := x_i V y_j V$ is locally finite. Pick $g_i, h_j \in C_c^r(G)$ such that $g_i, h_j \geq 0$, $K_i := \text{supp}(g_i) \subseteq x_i V$, $L_j := \text{supp}(h_j) \subseteq y_j V$, and $\|g_i\|_{L^1} = \|h_j\|_{L^1} = 1$. Pick $h_{i,j} \in C_c^r(G)$ such that $h_{i,j} \geq 0$, $\text{supp}(h_{i,j}) \subseteq V_{i,j}$ and $h_{i,j}|_{K_i L_j} = 1$. For $i, j \in \mathbb{N}$, let $p_{i,j}$ be a continuous seminorm on F . Let Z be the set of all $\gamma \in C_c^t(G, F)$ such that

$$(\forall i, j \in \mathbb{N}) \quad \|h_{i,j} \cdot \gamma\|_{p_{i,j}, L^1} < 1.$$

Lemma 1.3 entails that Z is an open 0-neighbourhood in $C_c^t(G, F)$. As β is assumed continuous, there exist open 0-neighbourhoods $X \subseteq C_c^r(G, E_1)$ and $Y \subseteq C_c^s(G, E_2)$ such that $\beta(X \times Y) \subseteq Z$. Using Lemma 1.15, for each $i \in \mathbb{N}$ we find a continuous seminorm p_i on E_1 such that $g_i \overline{B}_1^{p_i}(0) \subseteq X$. Likewise, for each $j \in \mathbb{N}$ there is a continuous seminorm q_j on E_2 such that $h_j \overline{B}_1^{q_j}(0) \subseteq Y$. For $v \in \overline{B}_1^{p_i}(0)$ and $w \in \overline{B}_1^{q_j}(0)$, we then have $\gamma := \beta(g_i v, h_j w) \in Z$, and thus $\|\gamma\|_{p_{i,j}, L^1} = \|h_{i,j} \cdot \gamma\|_{p_{i,j}, L^1} < 1$ (noting that $\text{supp}(\gamma) \subseteq K_i L_j$ on which $h_{i,j} = 1$). Therefore

$$\begin{aligned} 1 &> \|\gamma\|_{p_{i,j}, L^1} = \int_G p_{i,j}(\gamma(x)) d\lambda_G(x) \\ &= \int_G \int_G p_{i,j}(b(g_i(y)v, h_j(y^{-1}x)w)) d\lambda_G(y) d\lambda_G(x) \\ &= p_{i,j}(b(v, w)) \int_G \int_G g_i(y) h_j(y^{-1}x) d\lambda_G(y) d\lambda_G(x) \\ &= p_{i,j}(b(v, w)) \|g_i\|_{L^1} \|h_j\|_{L^1} = p_{i,j}(b(v, w)). \end{aligned}$$

Hence $b(\overline{B}_1^{p_i}(0) \times \overline{B}_1^{q_j}(0)) \subseteq \overline{B}_1^{p_{i,j}}(0)$, entailing that $p_{i,j}(b(x, y)) \leq p_i(x) q_j(y)$ for all $i, j \in \mathbb{N}$. Thus b satisfies product estimates.

6.3 Sufficiency of product estimates. As before, let (G, r, s, t, b) be as in the introduction, and assume that conditions (a) and (b) from Theorem A are satisfied. Also, assume that b satisfies product estimates (condition (c)). We show that β is continuous. To this end, let $(h_i)_{i \in \mathbb{N}}$ be a locally finite

⁶If G is discrete, these conditions are equivalent to countability of G .

partition of unity on G (smooth if G is a Lie group; continuous otherwise), with compact supports $K_i := \text{supp}(h_i)$. For all $i, j \in \mathbb{N}$, the convolution map $f_{i,j}: C_{K_i}^r(G, E_1) \times C_{K_j}^s(G, E_2) \rightarrow C_c^t(G, F)$ associated with b is then continuous (see Lemmas 2.1 and 2.2). We claim that the hypotheses of [22, Corollary 2.5] are satisfied. If this is true, then the bilinear map

$$f: \bigoplus_{i \in \mathbb{N}} C_{K_i}^r(G, E_1) \times \bigoplus_{j \in \mathbb{N}} C_{K_j}^s(G, E_2) \rightarrow C_c^t(G, F)$$

taking $((\gamma_i)_{i \in \mathbb{N}}, (\eta_j)_{j \in \mathbb{N}})$ to $\sum_{(i,j) \in \mathbb{N}^2} f_{i,j}(\gamma_i, \eta_j)$ is continuous (by the latter corollary). Since also the linear map $\Phi: C_c^r(G, E_1) \rightarrow \bigoplus_{i \in \mathbb{N}} C_{K_i}^r(G, E_1)$, $\gamma \mapsto (h_i \cdot \gamma)_{i \in \mathbb{N}}$ and the analogous map $\Psi: C_c^s(G, E_2) \rightarrow \bigoplus_{i \in \mathbb{N}} C_{K_i}^s(G, E_2)$ are continuous, we deduce that $\beta = f \circ (\Phi \times \Psi)$ is continuous.

To prove the claim, let $Q_{i,j}$ be continuous seminorms on $C_c^t(G, F)$ for all $i, j \in \mathbb{N}$. If $t < \infty$, choose $k, \ell \in \mathbb{N}_0$ with $k \leq r$, $\ell \leq s$ and $k + \ell = t$. If $t = r = s = \infty$, let $k := \ell := 0$. If $i < j$, then there exists a continuous seminorm $P_{i,j}$ on F and $s_{i,j} \in \mathbb{N}_0$ such that $s_{i,j} \leq s$ and

$$(\forall \gamma \in C_{K_i K_j}^t(G, F)) \quad Q_{i,j}(\gamma) \leq \|\gamma\|_{k, s_{i,j}, P_{i,j}}^{R,L}$$

(see Lemma 1.8). If $i \geq j$, there exists a continuous seminorm $P_{i,j}$ on F and $r_{i,j} \in \mathbb{N}_0$ such that $r_{i,j} \leq r$ and $Q_{i,j}(\gamma) \leq \|\gamma\|_{r_{i,j}, \ell, P_{i,j}}^{R,L}$ for all $\gamma \in C_{K_i K_j}^t(G, F)$. By hypothesis (c), there are continuous seminorms p_i on E_1 and q_j on E_2 such that $P_{i,j}(b(x, y)) \leq p_i(x)q_j(y)$ for all $i, j \in \mathbb{N}$ and all $x \in E_1$, $y \in E_2$. For $i, j \in \mathbb{N}$, let $P_i := \|\cdot\|_{k, p_i}^R$ and $Q_j := \|\cdot\|_{\ell, q_j}^L$. For $i, j \in \mathbb{N}$ with $i < j$, let $q_{i,j} := \lambda_G(K_i) \|\cdot\|_{s_{i,j}, q_j}^L$. Then

$$Q_{i,j}(\gamma *_{\mathfrak{b}} \eta) \leq \|\gamma *_{\mathfrak{b}} \eta\|_{k, s_{i,j}, P_{i,j}}^{R,L} \leq \|\gamma\|_{k, p_i}^R \|\eta\|_{s_{i,j}, q_j}^L \lambda_G(K_i) = P_i(\gamma) q_{i,j}(\eta) \quad (23)$$

for all $\gamma \in C_{K_i}^r(G, E_1)$ and $\eta \in C_{K_j}^s(G, E_2)$, using Lemma 2.6. If $i, j \in \mathbb{N}$ with $i \geq j$, let $p_{i,j} := \lambda_G(K_i) \|\cdot\|_{r_{i,j}, p_i}^R$. For $\gamma \in C_{K_i}^r(G, E_1)$ and $\eta \in C_{K_j}^s(G, E_2)$,

$$Q_{i,j}(\gamma *_{\mathfrak{b}} \eta) \leq \|\gamma *_{\mathfrak{b}} \eta\|_{r_{i,j}, \ell, P_{i,j}}^{R,L} \leq \|\gamma\|_{r_{i,j}, p_i}^R \|\eta\|_{\ell, q_j}^L \lambda_G(K_i) = p_{i,j}(\gamma) Q_j(\eta) \quad (24)$$

(using Lemma 2.6 again). By (23) and (24), the claim is established. \square

7 Convolution of C^r -maps and C_c^s -maps

Proposition 7.1 *Let (G, r, s, t, b) be as in the introduction, and*

$$\begin{aligned}\beta_b: C_c^r(G, E_1) \times C^s(G, E_2) &\rightarrow C^t(G, F) \text{ and} \\ \theta_b: C^r(G, E_1) \times C_c^s(G, E_2) &\rightarrow C^t(G, F)\end{aligned}$$

*be the convolution maps taking (γ, η) to $\gamma *_b \eta$. Then β_b and θ_b are hypocontinuous. The map β_b is continuous if and only if G is compact. Likewise, θ_b is continuous if and only if G is compact. Moreover, the convolution maps*

$$\begin{aligned}\beta_K: C_K^r(G, E_1) \times C^s(G, E_2) &\rightarrow C^t(G, F) \text{ and} \\ \theta_K: C^r(G, E_1) \times C_K^s(G, E_2) &\rightarrow C^t(G, F)\end{aligned}$$

*taking (γ, η) to $\gamma *_b \eta$ are continuous, for each compact subset $K \subseteq G$.*

Proof. Since $\theta_b(\gamma, \eta) = \beta_{b^\vee}(\eta^*, \gamma^*)^*$ for all $(\gamma, \eta) \in C^r(G, E_1) \times C_c^s(G, E_2)$ and $\beta_b(\gamma, \eta) = \theta_{b^\vee}(\eta^*, \gamma^*)^*$ for all $(\gamma, \eta) \in C_c^r(G, E_1) \times C^s(G, E_2)$, where the involutions denoted by $*$ are continuous linear maps (Lemma 1.13), the assertions concerning θ_b follow if we can establish those concerning $\beta := \beta_b$.

We first show that β_K is continuous for each compact subset $K \subseteq G$. To this end, recall that the topology on $C^t(G, F)$ is initial with respect to the linear maps $\rho_W: C^t(G, F) \rightarrow C^t(W, F)$, $\gamma \mapsto \gamma|_W$, for W ranging through the relatively compact, open subsets of G (cf. [20, Lemma 4.6]). Since $K^{-1}\overline{W}$ is compact, there exists $h \in C^\infty(G)$ with compact support $L := \text{supp}(h)$, such that $h|_{K^{-1}\overline{W}} = 1$. For $\gamma \in C_K^r(G, E_1)$ and $\eta \in C^s(G, E_2)$, we have $(\gamma *_b \eta)(x) = \int_G b(\gamma(x), \eta(y^{-1}x)) d\lambda_G(y) = \int_K b(\gamma(x), \eta(y^{-1}x)) d\lambda_G(y) = \int_K b(\gamma(x), (h \cdot \eta)(y^{-1}x)) d\lambda_G(y) = \int_G b(\gamma(x), (h \cdot \eta)(y^{-1}x)) d\lambda_G(y) = (\gamma *_b (h \cdot \eta))(x)$ for all $x \in W$ and hence

$$\rho_W(\beta_K(\gamma, \eta)) = \rho_W(\beta_K(\gamma, h \cdot \eta)) = \rho_W(\mu(\gamma, h \cdot \eta)),$$

using the convolution $\mu: C_K^r(G, E_1) \times C_L^s(G, E_2) \rightarrow C_{KL}^t(G, F) \subseteq C^t(G, F)$, which is continuous by Lemmas 2.1 and 2.2. Since also the multiplication operator $m_h: C^s(G, E_2) \rightarrow C_L^s(G, E_2)$, $\eta \mapsto h \cdot \eta$ is continuous (cf. [19, Lemma 3.9 and Proposition 3.10]), $\rho_W \circ \beta_K$ and hence β_K is continuous.

If $\gamma \in C_c^r(G, E_1)$, then $\beta_b(\gamma, \cdot) = \beta_{\text{supp}(\gamma)}(\gamma, \cdot): C^s(G, E_2) \rightarrow C^t(G, F)$ is continuous. For $\eta \in C^r(G, E_2)$, the map $\beta_b(\cdot, \eta): C_c^r(G, E_1) \rightarrow C^t(G, F)$ is linear and $\beta_b(\cdot, \eta)|_{C_K^r(G, E_1)} = \beta_K(\cdot, \eta): C_K^r(G, E_1) \rightarrow C^t(G, F)$ is continuous

for each compact set $K \subseteq G$. Hence, since $C_c^r(G, E_1) = \varinjlim C_K^r(G, E_1)$ as a locally convex space, $\beta_b(\cdot, \eta)$ is continuous. Thus β_b is separately continuous.

If $B \subseteq C_c^r(G, E_1)$ is a bounded set, then $B \subseteq C_K^r(G, E_1)$ for some compact set $K \subseteq G$ (Lemma 1.16 (c)), and thus $\beta|_{B \times C^s(G, E_2)} = \beta_K|_{B \times C^s(G, E_2)}$ is continuous. Hence β_b is hypocontinuous in the first argument (Remark B.1).

To see that β_b is hypocontinuous in the second argument, let $(\lambda_i)_{i \in I}$ be a family of linear maps $\lambda_i: F \rightarrow F_i$ to Banach spaces F_i , such that the topology on F is initial with respect to this family. Then the topology on $C^t(G, F)$ is initial with respect to the mappings $C^t(G, \lambda_i)$ for $i \in I$ (see [20, Lemma 4.14] for manifolds; cf. [15, 3.4.6] for topological spaces). Hence, by Lemma B.2 (c), we need only show that each of the maps $C^t(G, \lambda_i) \circ \beta_b = \beta_{\lambda_i \circ b}$ is hypocontinuous in the second argument. We may therefore assume now that F is a Banach space. Then there exist continuous linear mappings $\psi_1: E_1 \rightarrow F_1$ and $\psi_2: E_2 \rightarrow F_2$ to suitable Banach spaces F_1 and F_2 , and a continuous bilinear map $c: F_1 \times F_2 \rightarrow F$ such that $c \circ (\psi_1 \times \psi_2) = b$. Since $\beta_b = \beta_c \circ (C_c^r(G, \psi_1) \times C^s(G, \psi_2))$, we need only show that β_c is hypocontinuous (see Lemma B.2 (b)). We may therefore assume that all of E_1 , E_2 , and F are Banach spaces. Then $C_K^r(G, E_1)$ is a Fréchet space for each compact subset $K \subseteq G$, and hence barrelled. Hence also $C_c^r(G, E_1)$ is barrelled, like every locally convex direct limit of barrelled spaces [39, II.7.2]. As the first factor of its domain is barrelled, the separately continuous bilinear map $\beta_b: C_c^r(G, E_1) \times C^s(G, E_2) \rightarrow C^t(G, F)$ is hypocontinuous in the second argument [39, III.5.2]. As we already established its hypocontinuity in the first argument, β_b is hypocontinuous.

Finally, we show that β_b (and hence also θ_b) fails to be continuous if G is not compact. Pick $u \in E_1$, $v \in E_2$ such that $w := \beta(u, v) \neq 0$. Let $K \subseteq G$ be a compact identity neighbourhood and p be a continuous seminorm on F such that $p(w) > 0$. Then $W := \{\gamma \in C^t(G, F): \gamma(K) \subseteq B_1^p(0)\}$ is an open 0-neighbourhood in $C^t(G, F)$. To see that β_b is not continuous, let $U \subseteq C_c^r(G, E_1)$ and $V \subseteq C^s(G, E_2)$ be any 0-neighbourhoods. Let $(U_i)_{i \in I}$ be a locally finite cover of G by relatively compact, open sets. Since the topology on $C^s(G, E_2)$ is initial with respect to the restriction mappings $\rho_i: C^s(G, E_2) \rightarrow C^s(U_i, E_2)$, $\gamma \mapsto \gamma|_{U_i}$ (cf. [20, Lemma 4.12]), we find a finite subset $I_0 \subseteq I$ and 0-neighbourhoods $Q_i \subseteq C^s(U_i, E_2)$ for $i \in I_0$ such that

$$\bigcap_{i \in I_0} \rho_i^{-1}(Q_i) \subseteq V. \quad (25)$$

Since $K := \bigcup_{i \in I_0} \overline{U_i}$ is compact and we assume that G is not compact, the open set $G \setminus K$ is non-empty. We pick $\eta \in C_c^\infty(G)$ such that $\eta \geq 0$, $\eta \neq 0$ and $\text{supp}(\eta) \subseteq G \setminus K$. Then $\eta_a := a\gamma v \in V$ for each $a > 0$ (by (25)). Define $\gamma_a \in C_c^r(G, E_1)$ via $\gamma_a(x) := \frac{1}{a}\eta(x^{-1})u$. Since U is a 0-neighbourhood, there is $a_0 > 0$ such that $\gamma_a \in U$ for all $a \geq a_0$. Then

$$p((\gamma_a *_b \eta_{a^2})(1)) = ap(w) \int_G \eta(y^{-1})\eta(y^{-1}) d\lambda_G(y) = ap(w) \|\eta\|_{L^2}^2,$$

where the right hand side can be made > 1 for large a . Thus $\gamma_a *_b \eta_{a^2} \notin W$ although $\gamma_a \in U$ and $\eta_{a^2} \in V$. Hence $\beta_b(U \times V) \not\subseteq W$. Since U and V were arbitrary, β_b is not continuous. \square

A Background on vector-valued integrals

If E is a locally convex space, (X, Σ, μ) a measure space [7] and $\gamma: X \rightarrow E$ a function, we call a (necessarily unique) element $v \in E$ the *weak integral of γ with respect to μ* (and write $\int_X \gamma(x) d\mu(x) := v$) if $\lambda \circ \gamma: X \rightarrow \mathbb{R}$ is μ -integrable for each $\lambda \in E'$ and $\lambda(v) = \int_X \lambda(\gamma(x)) d\mu(x)$. If p is a continuous seminorm on E , using the Hahn-Banach Extension Theorem one finds that

$$p\left(\int_X \gamma(x) d\mu(x)\right) \leq \mu(X) \|\gamma\|_{p,\infty}. \quad (26)$$

Lemma A.1 *Let $(E, \|\cdot\|)$ be a locally convex space, X be a locally compact space, μ be a Borel measure on X (see 3.1) and $\gamma: X \rightarrow E$ be a continuous mapping with compact support K . If K is metrizable, assume that E is sequentially complete or satisfies the metric convex compactness property; if K is not metrizable, assume that E satisfies the convex compactness property. Then the weak integral $\int_X \gamma(x) d\mu(x)$ exists in E .*

Proof. See [29, 1.2.3] for the first case, [37, 3.27] for the two others. \square

The next two lemmas can be proved exactly as [9, Proposition 3.5].

Lemma A.2 *Let E be a locally convex space, X be a topological space, μ a Borel measure on a compact space K , and $f: X \times K \rightarrow E$ be a continuous map. Assume that the weak integral $g(x) := \int_K f(x, a) d\mu(a)$ exists for each $x \in X$. Then $g: X \rightarrow E$ is continuous.*

Lemma A.3 *In the situation of A.2, assume that $n \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty\}$, $X \subseteq \mathbb{R}^n$ is open, the partial derivatives $\partial_1^\alpha f(x, a)$ of f with respect to the variables in X exist for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, and define continuous maps $\partial_1^\alpha f: X \times K \rightarrow E$. Also, assume that the weak integrals $\int_K \partial_1^\alpha f(x, a) d\mu(a)$ exist in E for all α as before. Then $g: X \rightarrow E$, $x \mapsto \int_K f(x, a) d\mu(a)$ is C^r , and $\partial^\alpha g(x) = \int_K \partial_1^\alpha f(x, a) d\mu(a)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$ and $x \in K$.*

B Hypocontinuous bilinear maps

Hypocontinuity. As a special case of more general concepts, we call a bilinear map $\beta: E_1 \times E_2 \rightarrow F$ between locally convex spaces *hypocontinuous* if the following conditions are satisfied:

- (H1) For each 0-neighbourhood $V \subseteq F$ and bounded set $B_1 \subseteq E_1$, there exists a 0-neighbourhood $U \subseteq E_2$ such that $\beta(B_1 \times U) \subseteq V$; and:
- (H2) For each 0-neighbourhood $V \subseteq F$ and bounded set $B_2 \subseteq E_2$, there exists a 0-neighbourhood $U \subseteq E_1$ such that $\beta(U \times B_2) \subseteq V$.

In this case, β is separately continuous (as B_1, B_2 can be chosen as singletons). If β is separately continuous and satisfies (H1) (resp., (H2)), we say that β is hypocontinuous in its first (resp., its second) argument.

Remark B.1 It is well known that a separately continuous bilinear map $\beta: E_1 \times E_2 \rightarrow F$ between locally convex spaces is hypocontinuous in its second argument if and only if its restrictions $\beta|_{E_1 \times B}: E_1 \times B \rightarrow F$ are continuous for all bounded subsets $B \subseteq E_2$ (see, e.g., [21, Proposition 16.8]; cf. Proposition 4 in [12, Chapter III, §5, no. 3]).

Simple observations concerning hypocontinuous bilinear maps will be useful.

Lemma B.2 (a) *Let $\beta: E_1 \times E_2 \rightarrow F$ be a bilinear map between locally convex spaces which is hypocontinuous in its second argument. Let $\Lambda: F \rightarrow H$ be a continuous linear map to a locally convex space H . Then also $\Lambda \circ \beta: E_1 \times E_2 \rightarrow H$ is hypocontinuous in its second argument.*

- (b) *Let $\beta: E_1 \times E_2 \rightarrow F$ be a bilinear map between locally convex spaces which is hypocontinuous in its second argument. Let H_1, H_2 be locally convex spaces and $\psi_1: H_1 \rightarrow E_1$, $\psi_2: H_2 \rightarrow E_2$ be continuous linear maps. Then also $\beta \circ (\psi_1 \times \psi_2): H_1 \times H_2 \rightarrow F$ is hypocontinuous in its second argument.*

- (c) Let E_1, E_2 and F be locally convex spaces. If the topology on F is initial with respect to a family $(\Lambda_i)_{i \in I}$ of linear maps $\Lambda_i: F \rightarrow F_i$ to locally convex spaces F_i , then a bilinear map $\beta: E_1 \times E_2 \rightarrow F$ is hypocontinuous in its second argument if and only if $\Lambda_i \circ \beta$ is hypocontinuous in its second argument, for each $i \in I$.
- (d) Let $(E_i)_{i \in I}$ and $(F_j)_{j \in J}$ be families of locally convex spaces and let $\beta: (\bigoplus_{i \in I} E_i) \times (\bigoplus_{j \in J} F_j) \rightarrow H$ be a bilinear map to a locally convex space H . Then β is hypocontinuous in its second argument if and only if $\beta_{i,j} := \beta|_{E_i \times F_j}: E_i \times F_j \rightarrow H$ is hypocontinuous in its second argument, for all $(i, j) \in I \times J$.
- (e) If E_1, E_2 are locally convex spaces and $\beta: E_1 \times E_2 \rightarrow F$ is a continuous bilinear map to a Fréchet space F , then there exist continuous linear maps $\psi_1: E_1 \rightarrow F_1$ and $\psi_2: E_2 \rightarrow F_2$ to Fréchet spaces F_1, F_2 and a continuous bilinear map $\theta: F_1 \times F_2 \rightarrow F$ such that $\beta = \theta \circ (\psi_1 \times \psi_2)$.

Proof. (a) $\Lambda \circ \beta$ is separately continuous, and $\Lambda \circ \beta|_{E_1 \times B}$ is continuous for each bounded subset $B \subseteq E_2$. Hence $\Lambda \circ \beta$ is hypocontinuous in its second argument (see Remark B.1).

(b) The composition $\gamma := \beta \circ (\psi_1 \times \psi_2)$ is separately continuous. If $B \subseteq H_2$ is bounded, then $\psi_2(B)$ is bounded in E_2 , entailing that $\gamma|_{H_1 \times B} = \beta|_{E_1 \times \psi_2(B)} \circ \psi_1 \times (\psi_2|_B)$ is continuous. Hence γ is hypocontinuous in its second argument (using Remark B.1 again).

(c) If $x \in E_1$, then $\beta(x, \cdot): E_2 \rightarrow F$ is continuous if and only if $\Lambda_i \circ \beta(x, \cdot): E_2 \rightarrow F_i$ is continuous for each $i \in I$. Likewise, $\beta(\cdot, y)$ is continuous for $y \in E_2$ if and only if $\Lambda_i \circ \beta(\cdot, y)$ is continuous for each i . If $B \subseteq E_2$ is bounded, then $\beta|_{E_1 \times B}$ is continuous if and only if $\Lambda_i \circ \beta|_{E_1 \times B}$ is continuous for each $i \in I$. The assertion follows with Remark B.1.

(d) Write $E := \bigoplus_{i \in I} E_i$, $F := \bigoplus_{j \in J} F_j$. For $i \in I$, let $\lambda_i: E_i \rightarrow E$ be the usual embedding. Also, let $\mu_j: F_j \rightarrow F$ be the embedding for $j \in J$. If β is hypocontinuous in its second argument, then so is $\beta_{i,j} = \beta \circ (\lambda_i \times \mu_j)$, by (b).

Conversely, assume that each $\beta_{i,j}$ is hypocontinuous in its second argument. If $x = (x_i)_{i \in I} \in E$, then $x \in \bigoplus_{i \in I_0} E_i$ for some finite set $I_0 \subseteq I$. The linear map $\beta(x, \cdot): F \rightarrow H$ is continuous on F_j for each $j \in J$ (as it coincides with $\sum_{i \in I_0} \beta_{i,j}(x_i, \cdot)$ there). Hence $\beta(x, \cdot)$ is continuous (by the universal property of the locally convex direct sum). Likewise, $\beta(\cdot, y): E \rightarrow H$ is continuous for each $y \in F$, and thus β is separately continuous.

Let $B \subseteq \bigoplus_{j \in J} F_j$ be a bounded set and $U \subseteq H$ be an absolutely convex 0-neighbourhood. Then $B \subseteq \bigoplus_{j \in J_0} F_j =: X$ for some finite subset $J_0 \subseteq J$ [31, II.6.3]. Let n be the number of elements of J_0 . Excluding only trivial cases, we may assume that $n \geq 1$. For $j \in J_0$, let $\pi_j: X \rightarrow F_j$ be the projection onto F_j . Then $B_j := \pi_j(B)$ is bounded in F_j . For each $i \in I$, using the hypocontinuity of $\beta_{i,j}$ we now find a convex 0-neighbourhood $V_{i,j} \subseteq E_i$ such that $\beta_{i,j}(V_{i,j} \times B_j) \subseteq \frac{1}{n}U$. Set $V_i := \bigcap_{j \in J_0} V_{i,j}$. Then $\beta(V_i \times B_j) = \beta_{i,j}(V_i \times B_j) \subseteq \frac{1}{n}U$ and hence, using that $B \subseteq \sum_{j \in J_0} B_j$,

$$\beta(V_i \times B) \subseteq \sum_{j \in J_0} \beta_{i,j}(V_i \times B_j) \subseteq \sum_{j \in J_0} \frac{1}{n}U \subseteq U.$$

Now $V := \text{conv}(\bigcup_{i \in I} V_i)$ is a 0-neighbourhood in E . As every V_i is convex, for each $x \in V$ there are a finite set $I_0 \subseteq I$, elements $x_i \in V_i$ for $i \in I_0$, and $t_i \geq 0$ for $i \in I_0$ with $\sum_{i \in I_0} t_i = 1$ and $x = \sum_{i \in I_0} t_i x_i$. Hence, for each $y \in B$,

$$\beta(x, y) = \sum_{i \in I_0} t_i \beta(x_i, y) \in \sum_{i \in I_0} t_i U \subseteq U.$$

Thus $\beta(V \times B) \subseteq U$. Hence β is hypocontinuous in its second argument.

(e) Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of continuous seminorms on F defining its locally convex vector topology. For each $n \in \mathbb{N}$, we then find continuous seminorms P_n on E_1 and Q_n on E_2 such that $s_n(\beta(x, y)) \leq P_n(x)Q_n(y)$ for all $x \in E_1$, $y \in E_2$. Then $N_1 := \{x \in E_1 : (\forall n \in \mathbb{N}) P_n(x) = 0\}$ is a vector subspace of E_1 , and $N_2 := \{y \in E_2 : (\forall n \in \mathbb{N}) Q_n(y) = 0\}$ a vector subspace of E_2 . We equip E_1/N_1 with the vector topology defined by the sequence $(p_n)_{n \in \mathbb{N}}$ of seminorms given by $p_n(x + N_1) := P_n(x)$, and let F_1 be the completion of E_1/N_1 . Likewise, F_2 denotes the completion of E_2/N_2 , equipped with the seminorms q_n obtained from the Q_n . If $x \in N_1$ and $y \in E_2$, then $s_n(\beta(x, y)) = 0$ for each $n \in \mathbb{N}$ and thus $\beta(x, y) = 0$. Likewise, $\beta(x, y) = 0$ for all $x \in E_1$ and $y \in N_2$. As a consequence, $B: (E_1/N_1) \times (E_2/N_2) \rightarrow F$, $B(x+N_1, y+N_2) := \beta(x, y)$ is a well-defined bilinear map, which is continuous as $s_n(B(x+N_1, y+N_2)) = s_n(\beta(x, y)) \leq P_n(x)Q_n(y) = p_n(x+N_1)q_n(y+N_2)$. Since F is complete, B extends to a continuous bilinear map $\theta: F_1 \times F_2 \rightarrow F$ (cf. Theorem 1 in [13, III, §6, no. 5]). Let $\psi_1: E_1 \rightarrow F_1$ be the composition of inclusion map $E_1/N_1 \rightarrow F_1$ and the canonical map $E_1 \rightarrow E_1/N_1$. Define $\psi_2: E_2 \rightarrow F_2$ analogously. Then $\beta = \theta \circ (\psi_1 \times \psi_2)$ indeed. \square

C Proofs for Sections 1 to 3

Proof of Lemma 1.2. Since $\lambda := \oplus_{i \in I} \lambda_i$ is linear and continuous on each E_i , it is continuous (by the universal property of the locally convex direct sum). Moreover, λ is injective since each λ_i is injective. To see that λ is an embedding, let $U \subseteq \bigoplus_{i \in I} E_i =: E$ be a 0-neighbourhood. By Remark 1.1, there is a continuous seminorm p on E , of the form $p(x) = \sum_{i \in I} p_i(x_i)$ with continuous seminorms p_i on E_i , such that $B_1^p(0) \subseteq U$. Since λ_i is an embedding, there exists a continuous seminorm q_i on F_i such that $\lambda_i^{-1}(B_1^{q_i}(0)) \subseteq B_1^{p_i}(0)$. Then $q(y) := \sum_{i \in I} q_i(y_i)$ defines a continuous seminorm on $\bigoplus_{i \in I} F_i$. We now show that $\lambda(E) \cap B_1^q(0) \subseteq \lambda(B_1^p(0)) \subseteq \lambda(U)$ (whence $\lambda(U)$ is a 0-neighbourhood in $\lambda(E)$ and hence λ open onto its image – as required). In fact, $\lambda_i(E_i) \cap B_1^{q_i}(0) \subseteq \lambda_i(B_1^{p_i}(0))$. Hence $\lambda(E) \cap B_1^q(0) = \{y \in \bigoplus_{i \in I} \lambda_i(E_i) : \sum_{i \in I} q_i(y_i) < 1\} = \text{conv} \bigcup_{i \in I} (\lambda_i(E_i) \cap B_1^{q_i}(0)) \subseteq \text{conv} \bigcup_{i \in I} \lambda_i(B_1^{p_i}(0)) \subseteq \lambda(B_1^p(0))$. \square

Proof of Lemma 1.3. If $K \subseteq M$ is compact, then $F := \{j \in J : K \cap K_j \neq \emptyset\}$ is a finite set, and $\Phi(C_K^r(M, E)) \subseteq \bigoplus_{j \in F} C_{K_j}^r(M, E)$, identifying the right hand side with a topological vector subspace of $\bigoplus_{j \in J} C_{K_j}^r(M, E)$ in the usual way. Since $\bigoplus_{j \in F} C_{K_j}^r(M, E) \cong \prod_{j \in F} C_{K_j}^r(M, E)$ as a topological vector space, the restriction Φ_K of Φ to $C_K^r(M, E)$ will be continuous if we can show that all of its components with values in $C_{K_j}^r(M, E)$ are continuous, for $j \in F$. But these are the multiplication operators $C_K^r(M, E) \rightarrow C_{K_j}^r(M, E)$, $\gamma \mapsto h_j \cdot \gamma$, whose continuity is well-known (cf. [19, Proposition 3.10]).

If the h_j form a partition of unity, let $S : \bigoplus_{j \in J} C_{K_j}^r(M, E) \rightarrow C_c^r(M, E)$, $(\gamma_j)_{j \in J} \mapsto \sum_{j \in J} \gamma_j$ be the summation map, which is linear, and continuous because it is continuous on each summand. Then $S \circ \Phi = \text{id}_{C_c^r(M, E)}$. Hence Φ has a continuous left inverse and hence Φ is a topological embedding. \square

Proof of Lemma 1.4. Because $\text{supp}(\gamma|_S) \subseteq S \cap \text{supp}(\gamma)$ is compact for each $\gamma \in C_c^r(M, E)$, the map Φ makes sense, and it is clear that Φ is linear.

Φ is continuous: If $K \subseteq M$ is compact, there is a finite set $F \subseteq P$ such that $K \subseteq \bigcup_{S \in F} S$. Then $\Phi(C_K^r(M, E)) \subseteq \bigoplus_{S \in F} C_c^r(S, E)$, whence the restriction Φ_K of Φ to $C_K^r(M, E)$ will be continuous if we can show that all of its components with values in $C_c^r(S, E)$ are continuous, for $S \in F$. But these are the restriction maps $C_K^r(M, E) \rightarrow C_c^r(S, E)$, $\gamma \mapsto \gamma|_S$, and hence are continuous because they can be written as a composition of the continuous restriction map $C_K^r(M, E) \rightarrow C_{K \cap S}^r(S, E)$ (compare [19, Lemma 3.7]) and the continuous inclusion map $C_{K \cap S}^r(S, E) \rightarrow C_c^r(S, E)$.

If P is a partition of M into open sets, let $\Psi : \bigoplus_{S \in P} C_c^r(S, E) \rightarrow C_c^r(M, E)$

be the map taking a element $\eta := (\gamma_S)_{S \in P}$ of the left hand side to the function $\gamma \in C_c^r(M, E)$ defined piecewise via $\gamma(x) := \gamma_S(x)$ for $x \in S$. Then $\Phi(\Psi(\eta)) = \eta$, and thus Φ is surjective. Moreover, $\Psi(\Phi(\gamma)) = \gamma$ for $\gamma \in C_c^r(M, E)$, whence Φ is injective. Hence Φ is bijective, with $\Psi = \Phi^{-1}$. By the universal property of the locally convex direct sum, the linear map Ψ will be continuous if its restriction Ψ_S to the summand $C_c^r(S, E)$ is continuous, for each $S \in P$. To check this property, it suffices to show that the restriction Ψ_K of Ψ_S to $C_K^r(S, E)$ is continuous for each compact set $K \subseteq S$. But Ψ_K is continuous, as it is the composition of the map $C_K^r(S, E) \rightarrow C_K^r(M, E)$ extending functions by 0 off S (which is known to be continuous)⁷ and the continuous inclusion map $C_K^r(M, E) \rightarrow C_c^r(M, E)$. \square

Proof of Lemma 1.5. As the $C^{k+\ell}$ -property can be tested on the open cover of chart domains, we may assume that $M \subseteq \mathbb{R}^m$ is open. The proof is by induction on k . If $k = 0$, then γ is C^ℓ by hypothesis (and $\ell = k + \ell$). Now assume $k > 0$. Then γ is C^1 . For each $X \in \mathcal{F}_1$, the map $X.\gamma$ is C^{k-1} and $X_j \dots X_1.X.\gamma$ is C^ℓ for all $j \in \mathbb{N}_0$ such that $j \leq k-1$ and $X_i \in \mathcal{F}_{i+1}$ for $i \in \{1, \dots, j\}$. Hence $X.\gamma$ is $C^{k+\ell-1}$, by induction. Let $\mathcal{F}_1 = \{X_1, \dots, X_m\}$ and write $E_j = \frac{\partial}{\partial x_j}$ for $j \in \{1, \dots, m\}$. Then $E_j = \sum_{i=1}^m a_{i,j} X_i$ with smooth functions $a_{i,j} \in C^\infty(M)$ and thus $\frac{\partial \gamma}{\partial x_j} = \sum_{i=1}^m a_{i,j} (X_i.\gamma)$ is $C^{k+\ell-1}$. Since γ is C^1 and its first order partial derivatives are $C^{k+\ell-1}$, the map γ is $C^{k+\ell}$. \square

Proof of Lemma 1.6. Step 1. Let \mathcal{U} be the set of all open subsets of M which are diffeomorphic to open subsets of \mathbb{R}^m (where m is the dimension of M). For each $s \in \mathbb{N}_0 \cup \{\infty\}$, the topology on $C^s(M, E)$ is initial with respect to the restriction maps $\rho_U^s: C^s(M, E) \rightarrow C^s(U, E)$, $\gamma \mapsto \gamma|_U$ (see [20, Lemma 4.12]). Taking $s = r$, we deduce:⁸ If the lemma holds for each space $C^s(U, E)$, then \mathcal{O} on $C^r(M, E)$ is initial with respect to the maps

$$D_{X_j|_U, \dots, X_1|_U} \circ \rho_U^r = \rho_U^0 \circ D_{X_j, \dots, X_1}. \quad (27)$$

Taking $s = 0$, we deduce that $\mathcal{T}_{\mathcal{F}}$ is initial with respect to the maps on the right hand side of (27). Hence $\mathcal{O} = \mathcal{T}_{\mathcal{F}}$.

Step 2. In view of Step 1, it only remains to prove the lemma assuming that there exists a C^r -diffeomorphism $\phi: M \rightarrow V$ onto an open set $V \subseteq \mathbb{R}^m$.

⁷See [20, Lemma 4.24] if $r > 0$; the case $r = 0$ is elementary.

⁸If the topology on a space X is initial with respect to maps $f_i: X \rightarrow X_i$, with $i \in I$, and the topology on X_i is initial with respect to maps $g_{j,i}: X_i \rightarrow X_{j,i}$ to topological spaces $X_{j,i}$, for $j \in J_i$, then the topology on X is initial with respect to the maps $g_{j,i} \circ f_i$.

If X is a smooth vector field on M , let us write $X' := T\phi \circ X \circ \phi^{-1}$ for the corresponding vector field on V . Define $\Phi_s: C^s(M, E) \rightarrow C^s(V, E)$, $\gamma \mapsto \gamma \circ \phi^{-1}$ for $s \in \mathbb{N}_0 \cup \{\infty\}$. If $s \in \mathbb{N} \cup \{\infty\}$, then $\Phi_{s-1}(X \cdot \gamma) = X' \cdot \Phi_s(\gamma)$ for each vector field X on M and $\gamma \in C^s(M, E)$, whence

$$\Phi_0 \circ D_{X_j, \dots, X_1} = D_{X'_j, \dots, X'_1} \circ \Phi_r$$

for all $j \in \{0, \dots, r\}$ and $X_i \in \mathcal{F}_i$ for $i \in \{1, \dots, j\}$. Hence, since Φ_0 and Φ_r are isomorphisms of topological vector spaces (cf. [20, Lemma 4.11]), the topology on $C^r(M, E)$ is initial with respect to the D_{X_j, \dots, X_1} if and only if the topology on $C^r(V, E)$ is initial with respect to the $D_{X'_j, \dots, X'_1}$.

Step 3. By Step 2, we may assume that $M = V$ is an open subset of \mathbb{R}^m . We claim: If also $\mathcal{G} = (\mathcal{G}_1, \dots, \mathcal{G}_r)$ is an r -tuple of frames on V , then $\mathcal{T}_{\mathcal{G}} \subseteq \mathcal{T}_{\mathcal{F}}$. Hence also $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}_{\mathcal{G}}$ (reversing the roles of \mathcal{F} and \mathcal{G}) and thus $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}}$. But it is known that $\mathcal{O} = \mathcal{T}_{\mathcal{G}}$ if we choose $\mathcal{G}_i := \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ for all $i \in \{1, \dots, r\}$ (cf. [19, Proposition 4.4]). Thus $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}} = \mathcal{O}$, as required.

To establish the claim, recall that the multiplication operators

$$m_f: C^0(V, E) \rightarrow C^0(V, E), \quad m_f(\gamma) := f \cdot \gamma$$

are continuous for each $f \in C^0(V)$ (cf. [19, Lemma 3.9]). Write $\mathcal{F}_i = \{Y_{i,1}, \dots, Y_{i,m}\}$. Then each $X \in \mathcal{G}_i$ is a linear combination $X = \sum_{k=1}^m a_k Y_{i,k}$ with coefficients $a_k \in C^\infty(V)$. Hence, for all $j \in \{0, \dots, r\}$ and $X_1 \in \mathcal{G}_1, \dots, X_j \in \mathcal{G}_j$, it follows from the product rule that D_{X_j, \dots, X_1} can be written as a sum of operators of the form $m_{f_{k_i, \dots, k_1}} \circ D_{Y_{i,k_i}, \dots, Y_{1,k_1}}$, where $i \in \{0, \dots, j\}$, $k_1, \dots, k_i \in \{1, \dots, m\}$ and $f_{k_i, \dots, k_1} \in C^\infty(V)$. Since $\mathcal{T}_{\mathcal{F}}$ makes the map $D_{Y_{i,k_i}, \dots, Y_{1,k_1}}: C^r(V, E) \rightarrow C^0(V, E)$ continuous and also the multiplication operator $m_{f_{k_i, \dots, k_1}}: C^0(V, E) \rightarrow C^0(V, E)$ is continuous, it follows that $\mathcal{T}_{\mathcal{F}}$ makes $D_{X_j, \dots, X_1}: C^r(V, E) \rightarrow C^0(V, E)$ continuous. Hence $\mathcal{T}_{\mathcal{G}} \subseteq \mathcal{T}_{\mathcal{F}}$. \square

Proof of Lemma 1.8. By definition, the topology on $C_K^\infty(G, E)$ is initial with respect to the inclusion maps $C_K^\infty(G, E) \rightarrow C_K^n(G, E)$ with $n \in \mathbb{N}_0$. It hence suffices to prove the lemma for $t \in \mathbb{N}_0$. For (a), let $\mathcal{F}_i := \mathcal{F}_L$ for $i \in \{1, \dots, t\}$ (with notation from Definition 1.7). For the proof of (b), let $\mathcal{F}_i = \mathcal{F}_R$ for $i \in \{1, \dots, t\}$. In either case, let $\mathcal{F} := (\mathcal{F}_1, \dots, \mathcal{F}_t)$. Because the topology on $C_K^0(G, E)$ is defined by the seminorms $\|\cdot\|_{p, \infty}$, it follows with Lemma 1.6 that the topology on $C_K^t(G, E)$ is defined by the seminorms $\gamma \mapsto \|X_j \dots X_1 \cdot \gamma\|_{p, \infty}$ with $j \in \{0, \dots, t\}$ and $X_i \in \mathcal{F}_i$ for $i \in \{1, \dots, j\}$. The pointwise maximum of these (for fixed p) is $\|\cdot\|_{t,p}^L$ in (a), $\|\cdot\|_{t,p}^R$ in (b),

from which the descriptions in (a) and (b) follow. We now prove the first half of the final assertion (the second half can be shown analogously). If $i \in \{0, \dots, \ell\}$ and $j \in \{0, \dots, k\}$ are given, let $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_t)$ be the t -tupel whose first i entries are \mathcal{F}_R , followed by j entries \mathcal{F}_L , followed by $t-i-j$ arbitrary frames. Then Lemma 1.6 implies that $\gamma \mapsto \|X_{i+j} \dots X_1 \gamma\|_{\infty, p}$ is continuous on $C_K^t(G, E)$, for all $X_1 \in \mathcal{F}_1, \dots, X_{i+j} \in \mathcal{F}_{i+j}$. The maximum of all these seminorms for $i \leq \ell$ and $j \leq k$ is $\|\cdot\|_{k, \ell, p}^{L, R}$, which therefore is continuous. Hence, the topology defined by these seminorms is coarser than the given topology. On the other hand, taking $\mathcal{F}_i := \mathcal{F}_R$ for $i \in \{1, \dots, \ell\}$ and $\mathcal{F}_i = \mathcal{F}_L$ for $i \in \{\ell+1, \dots, t\}$, Lemma 1.6 shows that the topology on $C_K^t(G, E)$ is defined by the seminorms $\gamma \mapsto \|X_j \dots X_1 \gamma\|_{p, \infty}$, for $j \in \{0, \dots, t\}$, continuous seminorms p on E and $X_i \in \mathcal{F}_i$. For fixed p , each of the latter seminorms is bounded by $\|\cdot\|_{k, \ell, p}^{L, R}$. Hence the topology defined by the $\|\cdot\|_{k, \ell, p}^{L, R}$ is also finer than the given topology, and thus coincides with it. \square

Proof of Lemma 1.10. We discuss τ_g^L (τ_g^R can be treated analogously). Since left translation $L_g: G \rightarrow G$, $L_g(x) := gx$ is a C^r -diffeomorphism, the map $\Xi_g: C^r(G, E) \rightarrow C^r(G, E)$, $\gamma \mapsto \tau_g^L(\gamma) = \gamma \circ L_g$ is continuous and linear [19, Lemma 3.7]. Hence also its restriction $\Xi_{g, K}: C_K^r(G, E) \rightarrow C_{g^{-1}K}^r(G, E)$ is continuous, and so is the map $\Xi_{g, c}: C_c^r(G, E) \rightarrow C_c^r(G, E)$, $\gamma \mapsto \tau_g^L(\gamma)$ (as it is linear and its restriction $\Xi_{g, K}$ to $C_K^r(G, E)$ is continuous for each K). It is clear that each of the preceding maps is bijective; the inverse map is given by $\Xi_{g^{-1}}, \Xi_{g^{-1}, g^{-1}K}$ and $\Xi_{g^{-1}, c}$, respectively, and hence continuous. \square

Proof of Lemma 1.11. If X is a left invariant vector field on G and $\gamma \in C^1(G)$, then $(X.(\tau_g^L \gamma))(a) = d(\gamma \circ L_g)(X(a)) = d\gamma T(L_g)(X(a)) = d\gamma X(ga) = (X.\gamma)(ga)$ for $a \in G$ and thus $X.(\tau_g^L(\gamma)) = \tau_g^L(X.\gamma)$. Hence $\|X_j \dots X_1.(\tau_g^L(\gamma))\|_{g^{-1}K} = \|\tau_g^L(X_j \dots X_1.\gamma)\|_{g^{-1}K} = \|X_j \dots X_1.\gamma\|_K$ for all $j \in \{0, \dots, \ell\}$ and $X_1, \dots, X_j \in \mathcal{F}_L$ (using the notation from Definition 1.7). Now take the maximum over all j and X_1, \dots, X_j . \square

Proof of Lemma 1.13. The map $\eta_G: G \rightarrow G$, $x \mapsto x^{-1}$ is C^r . Hence $(\eta_G)^*: C^r(G, E) \rightarrow C^r(G, E)$, $\gamma \mapsto \gamma \circ \eta_G$ is continuous linear [19, Lemma 3.7]. As $f: G \rightarrow]0, \infty[$, $f(x) := \Delta_G(x^{-1})$ is C^r , we can consider the multiplication operator $m_f: C^r(G, E) \rightarrow C^r(G, E)$, $m_f(\gamma)(x) := f(x)\gamma(x)$, which is continuous linear (cf. [20, Proposition 4.16] if $r > 0$, and [19, Lemma 3.9] otherwise). Thus $\Theta = m_f \circ (\eta_G)^*$ is continuous linear. Because $\Phi \circ \Phi = \text{id}$, we deduce that Φ is an isomorphism of topological vector spaces. As a restriction of Θ , also the bijection Θ_K (with inverse $\Theta_{K^{-1}}$) is an isomorphism of topolog-

ical vector spaces. Finally, the linear map Φ_c is continuous (as its restrictions Φ_K to the spaces $C_K^r(G, E)$ are continuous) and hence an isomorphism of topological vector spaces (as $\Phi_c \circ \Phi_c = \text{id}$). \square

Proof of Lemma 1.14. Since Φ_v is linear, it will be continuous on $C_c^r(M) = \varinjlim C_K^r(M)$ if its restriction $\Phi_K: C_K^r(M) \rightarrow C_K^r(M, E) \subseteq C_c^r(M, E)$, $\gamma \mapsto \gamma v$ to $C_K^r(M)$ is continuous for each compact subset $K \subseteq M$. Let $\mu: \mathbb{R} \times E \rightarrow E$ be the scalar multiplication, and $h: M \rightarrow \mathbb{R}$ be a smooth map such that $L := \text{supp}(h)$ is compact and $h|_K = 1$. Because μ is smooth and $\mu(0, 0) = 0$, also the bilinear map $C_L^r(M, \mu): C_L^r(M) \times C_L^r(M, E) \cong C_L^r(M, \mathbb{R} \times E) \rightarrow C_L^r(E)$,

$$(\gamma, \eta) \mapsto \mu \circ (\gamma, \eta) = \gamma \eta$$

is smooth and hence continuous [19, Proposition 3.10]. Hence also $\Phi_K(\gamma) = \gamma v = h\gamma v = \mu(\gamma, hv)$ is continuous in γ . To complete the proof, pick $\lambda \in E'$ such that $\lambda(v) = 1$. Then $C_c^r(M, \lambda): C_c^r(M, E) \rightarrow C_c^r(M)$, $\gamma \mapsto \lambda \circ \gamma$ is a continuous linear map (by [19, Lemma 3.3] and the locally convex direct limit property), and $C_c^r(M, \lambda) \circ \Phi_v = \text{id}_{C_c^r(M)}$ because $\lambda \circ (\gamma v) = \gamma$. Since Φ_v has a continuous left inverse, it is a topological embedding. \square

Proof of Lemma 1.15. We first observe that the map $\Theta: E \rightarrow C^r(M, E)$ taking $v \in E$ to the constant map $\Theta(v): M \rightarrow E$, $x \mapsto v$ is continuous. In fact, the linear map $\Theta: E \rightarrow C^0(M, E)$ is continuous, as

$$(\forall v \in E) \quad \|\Theta(v)\|_{p,K} := \|\Theta(v)|_K\|_{p,\infty} \leq p(v)$$

for each continuous seminorm p on E and compact subset $K \subseteq M$. Since $d^k(\Theta(v)) = 0$ for all $k \in \mathbb{N}$ with $k \leq r$, we see that Θ is also continuous as a map to $C^r(M, E)$. We now use that $C^r(M, E)$ is a topological $C^r(M)$ -module under pointwise multiplication: Scalar multiplication $\mu: \mathbb{R} \times E \rightarrow E$ being continuous bilinear and hence smooth, also $C^r(M, \mu): C^r(M) \times C^r(M, E) \cong C^r(M, \mathbb{R} \times E) \rightarrow C^r(M, E)$, $(\gamma, \eta) \mapsto \mu \circ (\gamma, \eta) =: \gamma \eta$ is smooth (see [20, Proposition 4.16] if $r > 0$, [19, Lemma 3.9] if $r = 0$) and hence continuous. Thus $\Psi_E(\gamma, v) = \gamma \Theta(v) = C^r(M, \mu)(\gamma, \Theta(v))$ is continuous in (γ, v) . \square

Proof of Lemma 1.16. (a) It is clear from the definition of the topology that any 0-neighbourhood $U \subseteq C_K^r(M, E)$ contains an intersection $U_1 \cap \dots \cap U_n$ of 0-neighbourhoods of the form $U_i := \{\gamma \in C_K^r(M, E): \|d^{\ell_i} \gamma\|_{p_i, K_i} < \varepsilon_i\}$ with $n \in \mathbb{N}$, $\varepsilon_i > 0$, $\ell_i \in \mathbb{N}_0$ such that $\ell_i \leq r$, a continuous seminorm p_i on E and a compact set $K_i \subseteq T^{\ell_i} M$. Let $(\tilde{E}_p, \|\cdot\|_p)$ be the Banach space associated

to the continuous seminorm $p := p_1 + \dots + p_n$ on E , and $\lambda_p: E \rightarrow \tilde{E}_p$ be the canonical map. Then $V_i := \{\gamma \in C_K^r(M, \tilde{E}_p): \|d^{\ell_i} \gamma\|_{\|\cdot\|_p, K_i} < \varepsilon_i\}$ is an open 0-neighbourhood in $C_K^r(M, \tilde{E}_p)$ and hence also $V := V_1 \cap \dots \cap V_n$. Since $C_K^r(M, \lambda_p)^{-1}(V) \subseteq U$, the assertion follows.⁹

(b) Since M is σ -compact, there exists a locally finite C^r -partition of unity $(h_j)_{j \in \mathbb{N}}$ on M such that each h_j has compact support $K_j := \text{supp}(h_j)$ (take a partition of unity subordinate to a relatively compact open cover using Theorem 3.3 and Corollary 3.4 in [32, Chapter II] if $r > 0$, [15, Theorem 5.1.9] if $r = 0$). Then Φ from Lemma 1.3 is a topological embedding. Thus, for each 0-neighbourhood $U \subseteq C_c^r(M, E)$, there exist 0-neighbourhoods $U_j \subseteq C_{K_j}^r(M, E)$ such that $\Phi^{-1}(\bigoplus_{j \in \mathbb{N}} U_j) \subseteq U$. As a consequence of (a), each U_j contains a set of the form $C_{K_j}^r(M, \mu_j)^{-1}(V_j)$ for some Banach space E_j , continuous linear map $\mu_j: E \rightarrow E_j$, and 0-neighbourhood V_j in $C_{K_j}^r(M, E_j)$. Then $F := \prod_{j \in \mathbb{N}} E_j$ is a Fréchet space, and $\lambda := (\mu_j)_{j \in \mathbb{N}}: E \rightarrow F$ is continuous linear. Let $\pi_j: F \rightarrow E_j$ be the projection onto the j -th component, and $W_j := C_{K_j}^r(M, \pi_j)^{-1}(V_j)$. Because $\Psi: C_c^r(M, F) \rightarrow \bigoplus_{j \in \mathbb{N}} C_{K_j}^r(M, F)$, $\gamma \mapsto (h_j \gamma)_{j \in \mathbb{N}}$ is continuous linear (Lemma 1.3), the set $P := \Psi^{-1}(\bigoplus_{j \in \mathbb{N}} W_j)$ is a 0-neighbourhood in $C_c^r(M, F)$, and hence $Q := C_c^r(M, \lambda)^{-1}(P)$ is a 0-neighbourhood in $C_c^r(M, E)$ (using [19, Lemma 4.11]). If $\gamma \in Q$, then $h_j(\lambda \circ \gamma) \in W_j$ for each $j \in \mathbb{N}$, and hence $\pi_j \circ (h_j(\lambda \circ \gamma)) \in V_j$. Since $\pi_j \circ (h_j(\lambda \circ \gamma)) = h_j(\mu_j \circ \gamma) = \mu_j \circ (h_j \gamma)$, we deduce that $h_j \gamma \in U_j$ and thus $\gamma \in U$. Thus $Q \subseteq U$, and the assertion follows.

(c) Let $B \subseteq C_c^r(M, E)$ be bounded. As M is locally compact and paracompact, it admits a partition P into σ -compact open sets [15, Theorem 5.1.27]. Let $\Phi: C_c^r(M, E) \rightarrow \bigoplus_{S \in P} C_c^r(S, E)$ be as in Lemma 1.4. Then $\Phi(B)$ is bounded and hence $\Phi(B) \in \bigoplus_{S \in P_0} C_c^r(S, E)$ for a finite set $P_0 \subseteq P$. After replacing the $S \in P_0$ by their union, we may assume that $B \subseteq C_c^r(S, E)$ (as a consequence of Lemma 1.4, $C_c^r(S, E)$ can be regarded as a topological vector subspace of $C_c^r(M, E)$). Hence, we may assume that M is σ -compact. Let K_1, K_2, \dots be compact sets such that $M = \bigcup_{n=1}^{\infty} K_n$ and each $K_n \subseteq K_{n+1}^0$. Then $C_c^r(M, E)$ is the locally convex direct limit of $C_{K_1}^r(M, E) \subseteq C_{K_2}^r(M, E) \subseteq \dots$, where $C_{K_n}^r(M, E) = \{\gamma \in C_{K_{n+1}}^r(M, E): (\forall x \in M \setminus K_n) \gamma(x) = 0\}$ is a closed vector subspace of $C_{K_{n+1}}^r(M, E)$ and $C_{K_{n+1}}^r(M, E)$ induces the given topology of $C_{K_n}^r(M, E)$, for $n \in \mathbb{N}$. Hence B is a bounded set in $C_{K_n}^r(M, E)$ for an $n \in \mathbb{N}$ [39, II.6.5]. \square

⁹ $C_K^r(M, \lambda_p): C_K^r(M, E) \rightarrow C_K^r(M, \tilde{E}_p)$, $\gamma \mapsto \lambda_p \circ \gamma$ is also continuous [19, Lemma 3.3].

Proof of Lemma 2.1. If $(\gamma *_b \eta)(x) \neq 0$, then by (5) there is $y \in \text{supp}(\eta)$ such that $xy^{-1} \in \text{supp}(\gamma)$. Hence $x \in \text{supp}(\gamma)y \subseteq \text{supp}(\gamma)\text{supp}(\eta)$. Now assume that K is compact. Because the integrand in (6) is continuous as a map taking $(x, y) \in G \times \text{supp}(\gamma)$ to F , the continuity of $\gamma *_b \eta$ follows from Lemma A.2. If $M \subseteq G$ is compact and q a continuous seminorm on F , there are continuous seminorms p_1 on E_1 and p_2 on E_2 such that $q(b(v, w)) \leq p_1(v)p_2(w)$ for all $v \in E_1$, $w \in E_2$. For all $x \in M$, we infer $q((\gamma *_b \eta)(x)) \leq \int_K p_1(\gamma(y))p_2(\eta(y^{-1}x)) d\lambda_G(y) \leq \lambda_G(K)\|\gamma\|_{p_1, \infty}\|\eta|_{K^{-1}M}\|_{p_2, \infty}$. Thus

$$\|(\gamma *_b \eta)|_M\|_{q, \infty} \leq \lambda_G(K)\|\gamma\|_{p_1, \infty}\|\eta|_{K^{-1}M}\|_{p_2, \infty}, \quad (28)$$

and hence β is continuous.

If L is compact, we have $(\gamma *_b \eta)(x) = \int_L b(\gamma(xz^{-1}), \Delta_G(z^{-1})\eta(z)) d\lambda_G(z)$ by (8), from which continuity of $\gamma *_b \eta$ follows. Finally, β is continuous as $\|(\gamma *_b \eta)|_M\|_{q, \infty} \leq \lambda_G(L)\|\gamma|_{ML^{-1}}\|_{p_1, \infty}\|\eta\|_{p_2, \infty}\|\Delta_G|_{L^{-1}}\|_{\infty}$. \square

Proof of Proposition 2.2. We may assume $r, s \in \mathbb{N}_0$ and proceed by induction, starting with $r = 0$. If also $s = 0$, see Lemma 2.1.

Now let $s > 0$. If $x_0 \in G$, let $V \subseteq G$ be an open neighbourhood of x_0 with compact closure \overline{V} . If K is compact, set $M := K$. If K is not compact, then $M := \overline{V}L^{-1}$ is compact. In either case, $(\gamma *_b \eta)(x) = \int_M b(\gamma(y), \eta(y^{-1}x)) d\lambda_G(y)$ for all $x \in V$. Hence $(\gamma *_b \eta)|_V$ is C^s , by Lemma A.3, and thus $\gamma *_b \eta$ is C^s . The lemma also entails that

$$\begin{aligned} \mathcal{L}_{v_1} \cdot (\gamma *_b \eta)(x) &= \int_M b(\gamma(y), d\eta(T(L_{y^{-1}})(\mathcal{L}_{v_1}(x))) d\lambda_G(y) \\ &= \int_G b(\gamma(y), d\eta(T(L_{y^{-1}})T(L_x)(v_1)) d\lambda_G(y) \\ &= \int_G b(\gamma(y), (\mathcal{L}_{v_1} \cdot \eta)(y^{-1}x)) d\lambda_G(y) = (\gamma *_b (\mathcal{L}_{v_1} \cdot \eta))(x), \end{aligned}$$

using that $T(L_{y^{-1}})T(L_x)(v_1) = T(L_{y^{-1} \circ L_x})(v_1) = T(L_{y^{-1}x})(v_1) = \mathcal{L}_{v_1}(y^{-1}x)$. Since $\mathcal{L}_{v_1}\eta \in C_L^{s-1}(G, E_2)$, we obtain, by induction on s ,

$$\mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1} \cdot (\gamma *_b \eta) = \mathcal{L}_{v_i} \cdots \mathcal{L}_{v_2} \cdot (\gamma *_b \mathcal{L}_{v_1} \cdot \eta) = \gamma *_b (\mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1} \cdot \eta). \quad (29)$$

Now assume that $r > 0$. If $s = 0$, then (8) enables Lemma A.3 to be applied,¹⁰ and thus $\gamma *_b \eta \in C^r(G, F)$. Moreover, repeating the arguments

¹⁰For $x \in V$ as above, we can replace the domain of integration by a compact set again.

leading to (29) with right translations, we deduce from (8) that

$$\mathcal{R}_{w_j} \cdots \mathcal{R}_{w_1}(\gamma *_b \eta) = (\mathcal{R}_{w_j} \cdots \mathcal{R}_{w_1} \gamma) *_b \eta. \quad (30)$$

If γ is C^r and η is C^s , then $\mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1}(\gamma *_b \eta) = \gamma *_b (\mathcal{L}_{v_i} \cdots \mathcal{L}_{v_1} \eta)$ is C^r by the case $s = 0$, and (10) holds. Thus $\gamma *_b \eta$ is C^{r+s} , by Lemma 1.5. In view of Lemmas 1.6 and 2.1, the right hand side of (10) is continuous as a map $C_K^r(G, E_1) \times C_L^s(G, E_2) \rightarrow C_{KL}^0(G, F)$, for all v_1, \dots, v_i and w_1, \dots, w_j . Hence β is continuous as a map to $C_{KL}^{r+s}(G, F)$, by Lemma 1.6. \square

Proof of Lemma 2.4. Substituting $z = xy$ and using the left invariance of Haar measure, we obtain $(\gamma *_b \eta)^*(x) = \Delta_G(x^{-1})(\gamma *_b \eta)(x^{-1}) = \Delta_G(x^{-1}) \int_G b(\gamma(y), \eta(y^{-1}x^{-1})) d\lambda_G(y) = \Delta_G(x^{-1}) \int_G b(\gamma(x^{-1}z), \eta(z^{-1})) d\lambda_G(z) = \int_G b^\vee(\eta^*(z), \gamma^*(z^{-1}x)) d\lambda_G(z) = (\eta^* *_b^\vee \gamma^*)(x)$. \square

Proof of Lemma 2.5. (a) With $z = g^{-1}y$ and left invariance of λ_G , we get

$$\begin{aligned} (\tau_g^L(\gamma *_b \eta))(x) &= (\gamma *_b \eta)(gx) = \int_G b(\gamma(y), \eta(y^{-1}gx)) d\lambda_G(y) \\ &= \int_G b(\gamma(gz), \eta(z^{-1}x)) d\lambda_G(z) = ((\tau_g^L \gamma) *_b \eta)(x). \end{aligned}$$

(b) For $x \in G$, get $\tau_g^R(\gamma *_b \eta)(x) = (\gamma *_b \eta)(xg) = \int_G b(\gamma(y), \eta(y^{-1}xg)) d\lambda_G(y) = \int_G b(\gamma(y), \tau_g^R(\eta)(y^{-1}x)) d\lambda_G(y) = (\gamma *_b \tau_g^R(\eta))(x)$. \square

Proof of Lemma 2.6. Let \mathcal{F}_R and \mathcal{F}_L be as in Definition 1.7. If $i \in \{0, \dots, k\}$, $j \in \{0, \dots, \ell\}$, $X_1, \dots, X_i \in \mathcal{F}_R$ and $Y_1, \dots, Y_j \in \mathcal{F}_L$, then

$$\begin{aligned} \|X_1 \dots X_i Y_1 \dots Y_j (\gamma *_b \eta)\|_{q, \infty} &= \|(X_1 \dots X_i \gamma) *_b (Y_1 \dots Y_j \eta)\|_{q, \infty} \\ &\leq \|X_1 \dots X_i \gamma\|_{p_1, \infty} \|Y_1 \dots Y_j \eta\|_{p_2, \infty} \lambda_G(K) \leq \|\gamma\|_{k, p_1}^R \|\eta\|_{\ell, p_2}^L \lambda_G(K) \end{aligned}$$

by (10) and (28). All assertions now follow by passage to maxima over suitable i, j and the corresponding vector fields. \square

Proof of Proposition 2.7. Let us write β_b for β .

Step 1: Assume that G is σ -compact. Then the topology on $C_c^t(G, F)$ is initial with respect to maps of the form $C_c^t(G, \lambda_i)$ for certain continuous linear maps $\lambda_i: F \rightarrow F_i$ to Fréchet spaces (Lemma 1.16 (b)). Hence, by Lemma B.2 (c), β_b will be hypocontinuous if we can show that $C_c^t(G, \lambda_i) \circ \beta_b = \beta_{\lambda_i \circ b}$ is hypocontinuous for all $i \in I$. Thus, we may assume that F is a Fréchet space. Then $b = c \circ (\psi_1 \times \psi_2)$ with certain continuous linear maps $\psi_1: E_1 \rightarrow F_1$

and $\psi_2: E_2 \rightarrow F_2$ to Fréchet spaces and a continuous bilinear map $c: F_1 \times F_2 \rightarrow F$ (see Lemma B.2 (e)). Thus $\beta_b = \beta_c \circ (C_c^r(G, \psi_1) \times C_c^s(G, \psi_2))$, and we need only show that β_c is hypocontinuous (Lemma B.2 (b)). Hence E_1 and E_2 are Fréchet spaces, without loss of generality. Then $C_c^r(G, E_1)$ and $C_c^s(G, E_2)$ are locally convex direct limits of Fréchet spaces and hence barrelled [39, II.7.1 and II.7.2], whence β will be hypocontinuous if we can show that it is separately continuous (by [39, III.5.2]). For fixed $\eta \in C_c^s(G, E_2)$, let $L := \text{supp}(\eta)$. The map $\beta(\cdot, \eta): C_c^r(G, E_1) \rightarrow C_c^t(G, F)$ being linear, it will be continuous on $C_c^r(G, E_1) = \varinjlim C_K^r(G, E_1)$ if we can show that its restriction to $C_K^r(G, E_1)$ is continuous for each compact set $K \subseteq G$. But this is the case, since the convolution map $C_K^r(G, E_1) \times C_L^s(G, E_2) \rightarrow C_{KL}^t(G, F) \subseteq C_c^t(G, F)$ is continuous, by Lemmas 2.1 and 2.2. By an analogous argument, $\beta(\gamma, \cdot): C_c^s(G, E_2) \rightarrow C_c^t(G, F)$ is continuous for each $\gamma \in C_c^r(G, E_1)$.

Step 2. Let $H \subseteq G$ be a σ -compact open subgroup, $G/H := \{gH: g \in G\}$ be the set of left cosets and $H \backslash G := \{Hg: g \in G\}$ the set of right cosets. Since G/H is a partition of G into open sets, we can identify $C_c^r(G, E_1)$ with $\bigoplus_{M \in G/H} C_c^r(M, E_1)$, by Lemma 1.4. In particular, we can regard $C_c^r(M, E_1)$ as a topological vector subspace of $C_c^r(G, E_1)$ (extending functions by 0). Likewise, $C_c^s(G, E_2)$ can be identified with $\bigoplus_{N \in H \backslash G} C_c^s(N, E_2)$. By Lemma B.2 (d), β will be hypocontinuous if we can show that its restriction to $\beta_{M,N}: C_c^r(M, E_1) \times C_c^s(N, E_2) \rightarrow C_c^t(G, F)$ is hypocontinuous for all $M \in G/H$ and $N \in H \backslash G$. Write $M = mH$ and $N = Hn$ with suitable $m, n \in G$. Using Lemma 2.5, we can write

$$\beta_{M,N} = \tau_{m^{-1}}^L \circ \tau_{n^{-1}}^R \circ \beta_{H,H} \circ (\tau_m^L \times \tau_n^R), \quad (31)$$

where $\tau_m^L: C_c^r(M, E_1) \rightarrow C_c^r(H, E_1)$, $\tau_n^R: C_c^s(N, E_2) \rightarrow C_c^s(H, E_2)$, $\tau_{n^{-1}}^R: C_c^t(H, F) \rightarrow C_c^t(N, F)$ and $\tau_{m^{-1}}^L: C_c^t(N, F) \rightarrow C_c^t(mN, F) \subseteq C_c^t(G, F)$ are the respective translation maps, which are continuous as restrictions of translation maps on spaces of test functions on G (as in Lemma 1.10). Since $\beta_{H,H}: C_c^r(H, E_1) \times C_c^s(H, E_2) \rightarrow C_c^t(G, F)$ is hypocontinuous by Step 1, using Lemma B.2 (a) and (b) we deduce from (31) that also each of the maps $\beta_{M,N}$ is hypocontinuous. This completes the proof. \square

Proof of Lemma 3.2. As Φ is linear, it will be continuous if its restriction Φ_K to $M_K(X)$ is continuous for each compact set $K \subseteq X$. By the hypotheses, $\Phi(M_K(X))$ is contained in the finite direct sum $\bigoplus_{j \in J_K} M(A_j)$, whence Φ_K will be continuous if we can show that its components with values in $M(A_j)$

are continuous, for all $j \in J_K$. But these are continuous, as they have operator norm ≤ 1 (noting that $\|\mu|_{\mathcal{B}(A_j)}\| = |\mu|(A_j) \leq |\mu|(X) = \|\mu\|$). \square

Proof of Lemma 3.3. Φ is continuous: If $K \subseteq X$ is compact, there is $n \in \mathbb{N}$ such that $K \subseteq K_n$. Then $\Phi(M_K(X)) \subseteq \bigoplus_{j \leq n} M_{K_j}(X)$, whence the restriction Φ_K of Φ to $M_K(X)$ will be continuous if we can show that all of its components with values in $M_{K_j}(X)$ are continuous, for $j \in \{1, \dots, n\}$. But these take $\mu \in M_K(X)$ to $\mathbf{1}_{K_j \setminus K_{j-1}} \odot \mu$, and hence are continuous, as they have operator norm ≤ 1 (since $\|\mathbf{1}_{K_j \setminus K_{j-1}} \odot \mu\| = |\mathbf{1}_{K_j \setminus K_{j-1}} \odot \mu|(X) = (\mathbf{1}_{K_j \setminus K_{j-1}} \odot |\mu|)(X) = |\mu|(K_j \setminus K_{j-1}) \leq |\mu|(X) = \|\mu\|$.) Now consider the map $S: \bigoplus_{n \in \mathbb{N}} M_{K_n}(X) \rightarrow M_c(X)$, $(\mu_n)_{n \in \mathbb{N}} \mapsto \sum_{n=1}^{\infty} \mu_n$, which is continuous as it is continuous on each summand and linear. Then $S \circ \Phi = \text{id}_{M_c(X)}$. Thus Φ has a continuous left inverse, and hence Φ is a topological embedding. \square

Proof of Lemma 3.4. The linear map Φ will be continuous if its restriction Φ_K to $C_K(X)$ is continuous for each compact set $K \subseteq X$. The latter holds, since $\|\Phi_K(\gamma)\| = \|\gamma \odot \mu\| = (|\gamma| \odot \mu)(X) = \|\gamma\|_{L^1} \leq \mu(K) \|\gamma\|_{\infty}$. Likewise, the restriction Ψ_K of Ψ to $L_K^1(X)$ is continuous because $\|\Psi_K(\gamma)\| \leq \|\gamma\|_{L^1}$. \square

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